Section 12.4: Linear Functions

What is a Linear Function of Two Variables?

In two-variable calculus, a *linear function* is one whose graph is a plane.

Why is the graph of a linear function a plane?

The defining characteristic of a linear function of one variable was that they had a constant slope. If we are standing on a plane, the slope depends on the direction we walk. But in any given direction, the slope will remain constant as we walk across the plane. Most notably, this is true if we walk in the direction of the postive x-axis or positive y-axis. Therefore, on a plane, the slope ratios $\Delta z/\Delta x$ (with y fixed) and $\Delta z/\Delta y$ (with x fixed) remain constant. Thus we arrive at the following definition for the functions defining a plane:

If a *plane* has slope m in the x-direction, has slope n in the y-direction, and passes through the point (x_0, y_0, z_0), then its equation is

$$z = z_0 + m(x - x_0) + n(y - y_0).$$

The plane is the graph of the *linear function*

$$f(x, y) = z_0 + m(x - x_0) + n(y - y_0).$$

If we write $c = z_0 - mx_0 - ny_0$, then we can write $f(x, y)$ in the equivalent form

$$f(x, y) = mx + ny + c.$$

Any plane in $\mathbb{R}^3$ can be uniquely determined by a three points, so long as they do not lie on the same line.

Examples:

1. Find the linear function whose graph is the plane through the points $(4, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 2)$.

From a Numerical Point of View

Consider the linear function $f(x, y)$ represented by the following table:

$x\backslash y$	4	6	8	10	12
5	3	6	9	12	15
10	7	10	13	16	19
15	11	14	17	20	23
20	15	18	21	24	27
25	19	22	25	28	31

One will notice that the values of z given by the above table increase linearly in both the x and y directions. The value of z jumps by 4 units for each 5 units traveled in the direction of increasing x with y fixed, and jumps by 3 units for each 2 units traveled in the direction of increasing y with x fixed. Therefore, we can conclude that the slope in the x-direction is $m = \Delta x/\Delta y$ (y fixed) $= 4/5$, and that the slope in the y-direction is $\Delta z/\Delta y$ (x fixed) $= 3/2$.

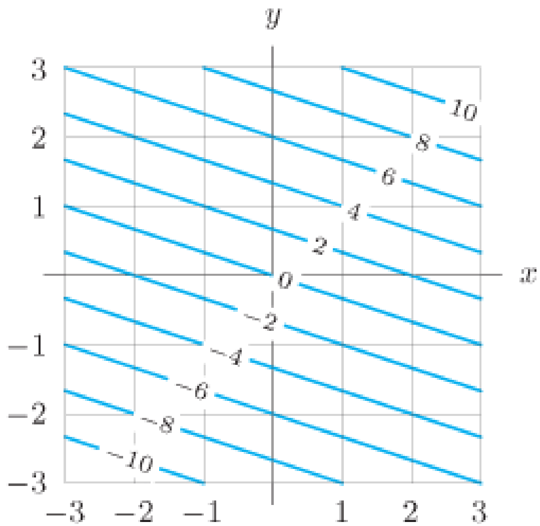
Examples:

2. Find an equation for the linear function with the given values.

$x\backslash y$	10	20	30	40
100	3	6	9	12
200	2	5	8	11
300	1	4	7	10
400	0	3	6	9

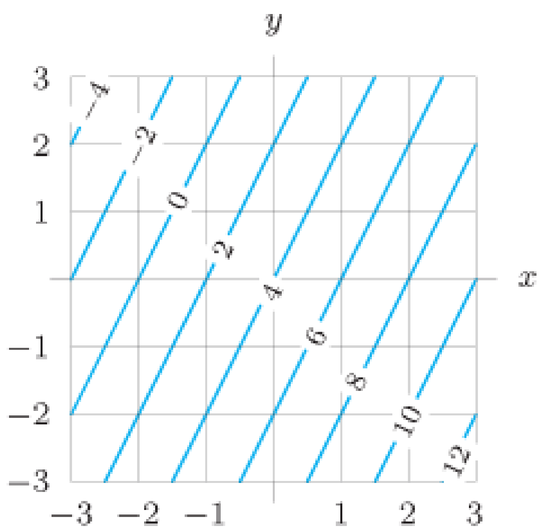
What Does the Contour Diagram of a Linear Function Look Like?

Pictured below is a contour diagram for a linear function $f(x, y)$. Discuss the properties of this contour diagram.



Examples:

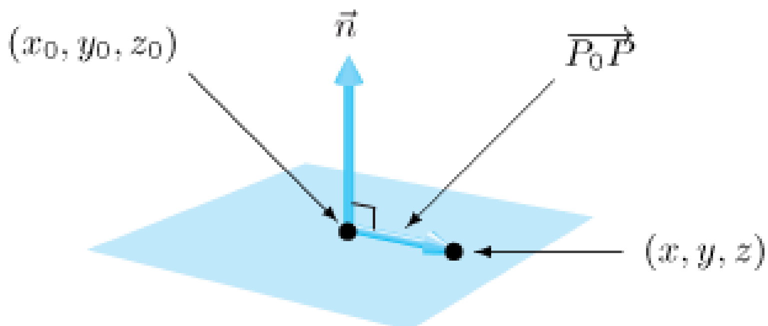
3. Find an equation for the linear function given by the following contour diagram.



Normal Vectors and the Equation of a Plane

It is also possible to utilize any vector perpendicular to a plane, which we will refer to as a *normal vector*, and a point P_0 on the plane to write down an equation for the plane.

Let $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ be a normal vector to the plane, let $P_0 = (x_0, y_0, z_0)$ be a fixed point on the plane, and let $P = (x, y, z)$ be any other point on the plane. Use the figure below to come up with an equation for the plane.



An *equation of the plane* with normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ and containing the point $P_0 = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Letting $d = ax_0 + by_0 + cz_0$ (a constant), we can write the equation of the plane in the form

$$ax + by + cz = d.$$

Examples:

4. Find an equation of a plane that passes through the point $(2, -1, 3)$ and is perpendicular to the vector $\vec{n} = 5\vec{i} + 4\vec{j} - \vec{k}$.

5. A plane has equation $z = 5x - 2y + 7$. Find a value of λ making the vector $\lambda\vec{i} + \vec{j} + 0.5\vec{k}$ normal to the plane.

6. Let $A = (-1, 3, 0)$, $B = (3, 2, 4)$ and $C = (1, -1, 5)$. Find an equation for the plane that passes through these three points.

7. Find a vector parallel to the line of intersection of the two planes $4x - 3y + 2z = 12$ and $x + 5y - z = 25$.