Section 13.3: The Dot Product

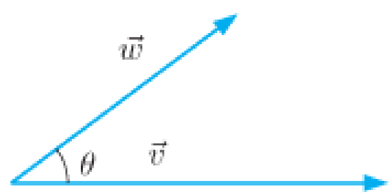
Definition of the Dot Product

The dot product gives us a way of "multiplying" two vectors and ending up with a scalar quantity. It can give us a way of computing the angle formed between two vectors. In the following definitions, assume that $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and that $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

The following two definitions of the **dot product**, or **scalar product**, $\vec{v} \cdot \vec{w}$, are equivalent:

- **Geometric definition**
$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta,$$
where θ is the angle between $\vec{v}$ and $\vec{w}$ and $0 \leq \theta \leq \pi$.
- **Algebraic definition**
$$\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3.$$
Notice that the dot product of two vectors is a *number*, not a vector.

The figure below depicts what is meant when we refer to the angle between two vectors:



A common theme throughout this course will be that many objects will be given two equivalent definitions: a geometric definition and an algebraic definition. It is usually the case that the geometric definition helps us gain insight into what the object is, while the algebraic definition gives us a way to compute the object.

It might not be clear that these two definitions are, in fact, equivalent.

Examples:

1. Suppose $\vec{v} = \vec{j}$ and $\vec{w} = \vec{i} + \vec{j}$. Compute $\vec{v} \cdot \vec{w}$ both geometrically and algebraically.

Properties of the Dot Product

Each of these properties is relatively easy to justify

PROPERTIES OF THE DOT PRODUCT: For any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ and any scalar λ,

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
2. $\vec{v} \cdot (\lambda \vec{w}) = \lambda(\vec{v} \cdot \vec{w}) = (\lambda \vec{v}) \cdot \vec{w}$
3. $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$

Perpendicularity, Magnitude, and Dot Products

We are all aware that to lines are perpendicular if and only if they intersect at an angle of $\pi/2$, or $90°$. The perpendicularity of two vectors is defined similarly: two vectors are perpendicular if the angle between them is $\pi/2$ ($90°$). Since the dot product between two vectors $\vec{v}$ and $\vec{w}$ is given by $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$, the dot product gives us a convenient way of characterizing perpendicularity:

Two non-zero vectors $\vec{v}$ and $\vec{w}$ are **perpendicular**, or **orthogonal**, if and only if

$$\vec{v} \cdot \vec{w} = 0$$

Magnitude and dot product are related as follows:

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2.$$

Question: If $\vec{v}$ and $\vec{w}$ are any two vectors in $\mathbb{R}^3$, what are the maximum and minimum values of $\vec{v} \cdot \vec{w}$?

Examples:

1. Compute the angle between the vectors $\vec{i} + \vec{j} - \vec{k}$ and $2\vec{i} + 3\vec{j} + \vec{k}$.

2. Which pairs (if any) of vectors from the following list
   (a) Are perpendicular?
   (b) Are parallel?
   (c) Have an angle less than $\pi/2$ between them?
   (d) Have an angle more than $\pi/2$ between them?

$$\vec{a} = \vec{i} - 3\vec{j} - \vec{k}, \qquad \vec{b} = \vec{i} + \vec{j} + 2\vec{k}$$

$$\vec{c} = -2\vec{i} - \vec{j} + \vec{k}, \qquad \vec{d} = -\vec{i} - \vec{j} + \vec{k}$$

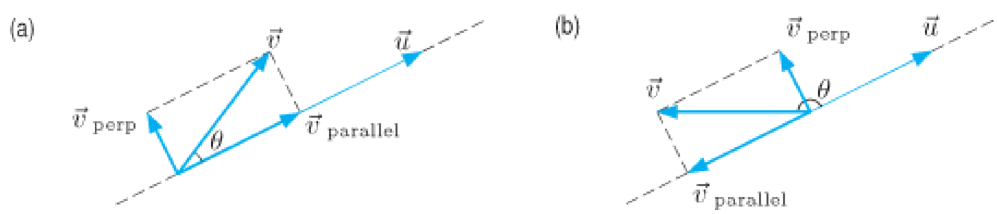
The Dot Product in n Dimensions

> If $\vec{u} = \langle u_1, u_2, \ldots, u_n\rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n\rangle$, then the dot product of $\vec{u}$ and $\vec{v}$ is the *scalar*
> $$\vec{u}\cdot\vec{v} = \sum_{j=1}^{n} u_j v_j = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Resolving a Vector into Components: Projections

We have already seen how to resolve a vector $\vec{v}$ into components that point in the directions of $\vec{i}$, $\vec{j}$, and $\vec{k}$. There is no reason that we shouldn't be able to resolve a vector into components that point in any directions we desire. After all, there is nothing fundamentally special about the placement of the x, y, and z-axes.

Suppose we have a unit vector, $\vec{u}$, and we want to resolve a vector $\vec{v}$ into components that point in a direction parallel to $\vec{u}$ and perpendicular to $\vec{u}$. The vector component of $\vec{v}$ that points in a direction parallel to $\vec{u}$ will be referred to as the *projection* of $\vec{v}$ onto $\vec{u}$, and we will denote it by $\vec{v}_{\text{parallel}}$. The component of $\vec{v}$ in the direction perpendicular to $\vec{u}$ will be denoted by $\vec{v}_{\text{perp}}$.



Problem: Using the geometry of the figures above, try to come up with a formula for $\vec{v}_{\text{parallel}}$

Projection of $\vec{\mathbf{v}}$ on the Line in the Direction of the Unit Vector $\vec{\mathbf{u}}$

If $\vec{v}_{\text{parallel}}$ and $\vec{v}_{\text{perp}}$ are components of $\vec{v}$ that are parallel and perpendicular, respectively, to the unit vector $\vec{u}$, then

$$\vec{v}_{\text{parallel}} = (\vec{v} \cdot \vec{u})\,\vec{u},$$

and since $\vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\text{perp}}$, we have

$$\vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}}.$$

Examples:

3. Write $\vec{a} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular, to $\vec{d} = 2\vec{i} - 4\vec{j} + \vec{k}$

A Physical Interpretation of the Dot Product: Work

You might recall that if a force of magnitude F acts on an object through a displacement d, then the *work* done on that object by F is given by

$$W = Fd,$$

provided that the force and the displacement are in the same direction. If the force and the displacement do not point in the direction, the dot product gives us a convenient way of calculating the work done by the force.

It is easiest to see what is going to happen if we assume that the angle θ, between the force $\vec{F}$ and the displacement $\vec{d}$ is between 0 and $\pi/2$. In this case, we can resolve $\vec{F}$ into components which point parallel and perpendicular to $\vec{d}$:

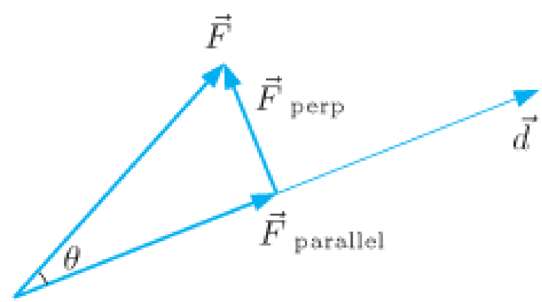
$$\vec{F} = \vec{F}_{\text{parallel}} + \vec{F}_{\text{perp}}.$$

Then the work done by $\vec{F}$ is given by

$$W = \|\vec{F}_{\text{parallel}}\| \, \|\vec{d}\|.$$

But it is easy to see, using the figure below, that this means that

$$W = (\|\vec{F}\| \cos\theta)\|\vec{d}\| = \|\vec{F}\| \, \|\vec{d}\| \cos\theta = \vec{F} \cdot \vec{d}.$$



The *work*, W, done by a force $\vec{F}$ acting on an object through a displacement $\vec{d}$ is given by

$$W = \vec{F} \cdot \vec{d}.$$

Examples:

4. Given $\vec{v} = 3\vec{i} + 4\vec{j}$ and force vector $\vec{F} = -3\vec{i} - 5\vec{j}$, fin

    (a) The component of $\vec{F}$ parallel to $\vec{v}$.

    (b) The component of $\vec{F}$ perpendicular to $\vec{v}$.

    (c) The work, W, done by force $\vec{F}$ through displacement $\vec{v}$.

5. A canoe is moving with velocity $\vec{v} = 5\vec{i} + 3\vec{j}$ m/s relative to the water. The velocity of the current in the water is $\vec{c} = \vec{i} + 2\vec{j}$ m/s.

    (a) What is the speed of the current?

    (b) What is the speed of the current in the direction of the canoe's motion?