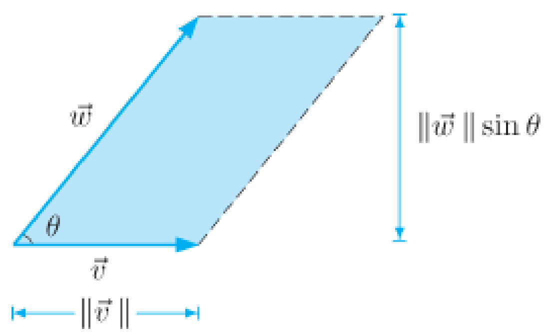
Section 13.4: The Cross Product

In the previous section, we learned of a way to "multiply" two vectors to get a scalar quantity. In this section, we will learn a different way to multiply two vectors in which the result is another vector, which we refer to as the *cross product*.

The Area of a Parallelogram

Let us begin by talking about the area of the parallelogram formed by the vectors $\vec{v}$ and $\vec{w}$ with an angle of θ between them.



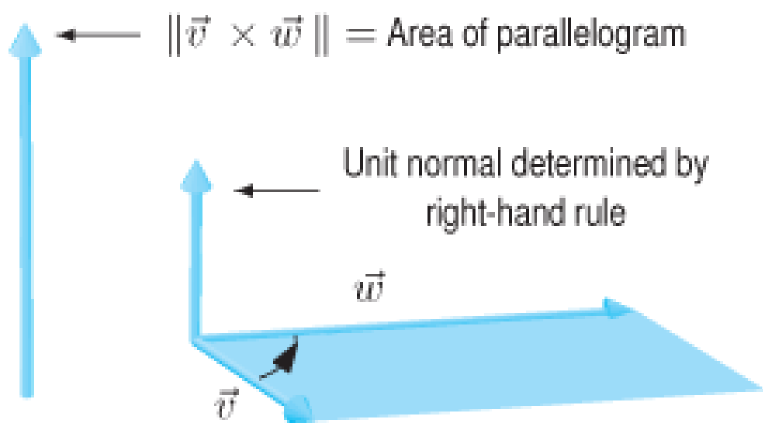
The area of the figure shown above can be seen to be given by

$$\text{Area of Parallelogram} = \text{Base} \times \text{Height} = \|\vec{v}\|\,\|\vec{w}\|\sin\theta$$

Definition of the Cross Product

As we did with the dot product in the previous section, we will give two equivalent definitions for the cross product. The first will be a geometric definition and the last will be an algebraic definition. The geometric definition will require us to go into some detail describing an important mathematical convention that is often used to give us some consistency in our definitions.

We are going to define the cross product of $\vec{v}$ and $\vec{w}$ to be a vector that points perpendicular to both $\vec{v}$ and $\vec{w}$, and we can describe this direction by assigning a *unit normal* vector $\vec{n}$ that will point in the direction of $\vec{v} \times \vec{w}$. But since there are two directions that are perpendicular to both $\vec{v}$ and $\vec{w}$, we need to resolve this ambiguity by introducing the *right-hand rule*:



THE RIGHT-HAND RULE: Place $\vec{v}$ and $\vec{w}$ so that their tails coincide and curl the fingers of your right hand through the smaller of the two angles formed from $\vec{v}$ to $\vec{w}$; your thumb points in the direction of the unit normal vector $\vec{n}$.

The following two definitions of the *cross product* or *vector product* $\vec{v} \times \vec{w}$ are equivalent:

- GEOMETRIC DEFINITION: If $\vec{v}$ and $\vec{w}$ are not parallel, then the cross product is the vector with magnitude equal to the area of the parallelogram with edges $\vec{v}$ and $\vec{w}$ pointing in the direction of $\vec{n}$:
$$\vec{v} \times \vec{w} = (\|\vec{v}\|\|\vec{w}\| \sin\theta)\,\vec{n},$$
where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$ and $\vec{n}$ is the unit vector perpendicular to $\vec{v}$ and $\vec{w}$ (*unit normal*) pointing in the direction given by the right-hand rule. If $\vec{v}$ and $\vec{w}$ are parallel, then $\vec{v} \times \vec{w} = \vec{0}$.

- ALGEBRAIC DEFINITION:
$$\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k},$$
where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

The most convenient way to remember how to compute the cross product using the algebraic definition is as the determinant of a 3×3 matrix whose first row consists of the unit vector $\vec{i}$, $\vec{j}$, and $\vec{k}$.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - v_3w_2)\vec{i} - (v_1w_3 - v_3w_1)\vec{j} + (v_1w_2 - v_2w_1)\vec{k},$$

Let us try our hand at computing some cross products.

Examples:

1. Use the algebraic definition of the cross product to compute $\vec{v} \times \vec{w}$.

   (a) $\vec{v} = \vec{i} + \vec{j} + \vec{k}$, $\vec{w} = \vec{i} + \vec{j} - \vec{k}$

   (b) $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{w} = \vec{i} + 2\vec{j} - \vec{k}$

2. Use the geometric definition of the cross product to find $\vec{v} \times \vec{w}$

   (a) $\vec{v} = 2\vec{i}$, $\vec{w} = \vec{i} + \vec{j}$

   (b) $\vec{v} = \vec{i} + \vec{j}$, $\vec{w} = \vec{i} - \vec{j}$

Properties of the Cross Product For vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and scalar λ,

1. $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$
2. $(\lambda\vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \times (\lambda\vec{w})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

Examples:

3. For $\vec{a} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$, find $\vec{a} \times \vec{b}$ and check that it is perpendicular to both $\vec{a}$ and $\vec{b}$

4. If $\vec{v} \times \vec{w} = 2\vec{i} - 3\vec{j} + 5\vec{k}$, and $\vec{v} \cdot \vec{w} = 3$, find $\tan\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$

Areas and Volumes Using the Cross Product and Determinants

The geometric definition of the cross product $\vec{v} \times \vec{w}$ gives us a nice way of computing the areas of parallelograms. $\vec{v} \times \vec{w}$ is defined to be a vector whose magnitude is equal to the area of the parallelogram with edges $\vec{v}$ and $\vec{w}$ if they are placed in such a way that their tails coincide. Therefore, we arrive at the following results:

AREA OF A PARALLELOGRAM: The area of the parallelogram with edges $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ is given by

$$\text{Area} = \|\vec{v} \times \vec{w}\|,$$

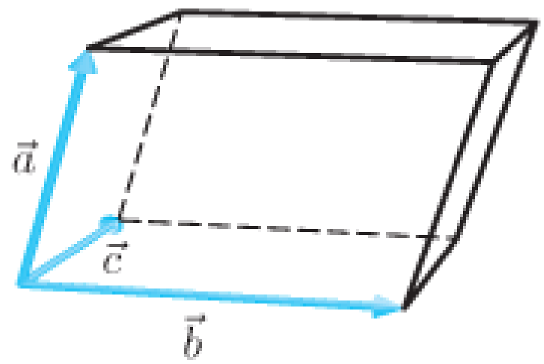
where

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Examples:

5. Find the area of a parallelogram with vertices $P_1 = (0, 0, 0)$, $P_2 = (2, 4, 2)$, $P_3 = (3, 0, 0)$, and $P_4 = (5, 4, 2)$.

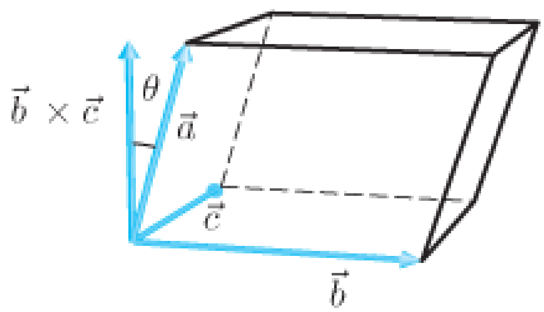
Consider the parallelepiped formed by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, shown below.



It is clear from the image that the base of the parallelepiped is itself a parallelogram. In fact, it is the parallelogram with edges $\vec{b}$ and $\vec{c}$, and therefore we have

$$\text{Area of base of parallelepiped} = \|\vec{b} \times \vec{c}\|.$$

The height of the parallelepiped can then be seen by considering the figure below:



The height of the parallelogram is given by $\|a\| \cos\theta$, where θ is the angle between $\vec{b} \times \vec{c}$ and $\vec{a}$. Thus, we have the following result:

The volume of the parallelepiped with edges $\vec{a}$, $\vec{b}$, and $\vec{c}$ is given by

$$\text{Volume of parallelepiped} = \text{Base} \cdot \text{Height} = \|\vec{b} \times \vec{c}\|\|\vec{a}\| \cos\theta = (\vec{b} \times \vec{c}) \cdot \vec{a}.$$