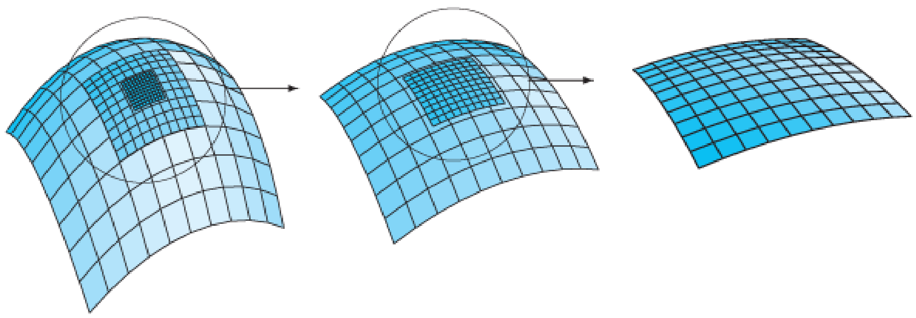
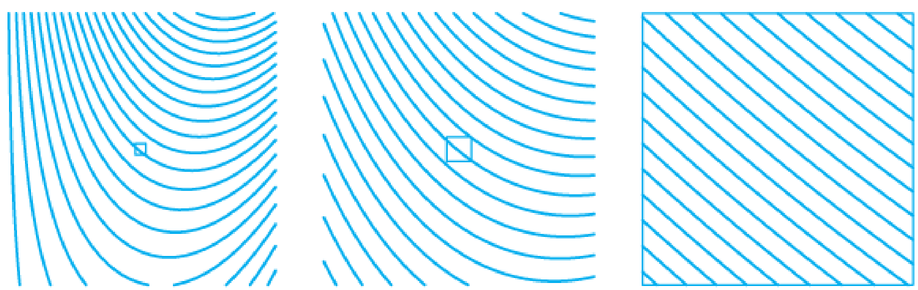
Section 14.3 Local Linearity and the Differential

The concept behind local linearity is pretty intuitive. The more we zoom in on the graph of a 2-variable function, the more the graph begins to resemble a plane.





Zooming In Algebraically: Differentiability

By zooming in on the graph of a two-variable function $f(x, y)$ near a point (a, b), we see that $f(x, y)$ is well approximated by a linear function, $L(x, y)$:

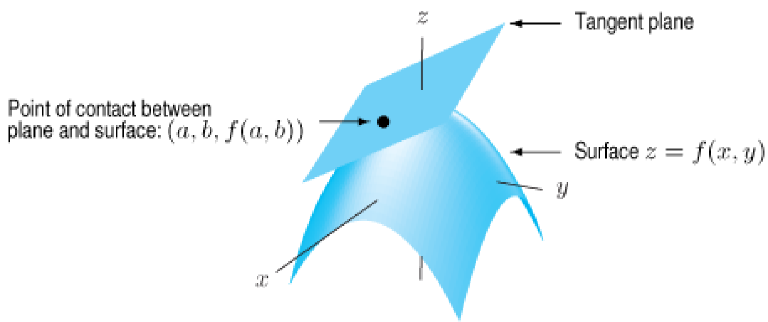
$$f(x, y) \approx L(x, y).$$

The graph of $z = L(x, y)$ is nothing more than the graph of the tangent plane to $f(x, y)$ at the point (a, b).

The Tangent Plane

Assuming that f is differentiable at (a, b), the equation of the tangent plane is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$



Examples:

1. Find an equation of the tangent plane to $z = \sin(xy)$ at the point $(2, 3\pi/4)$.

2. Find an equation of the tangent plane to the surface given by $x^2y + \ln(xy) + z = 6$ at the point $(4, 0.25, 2)$.

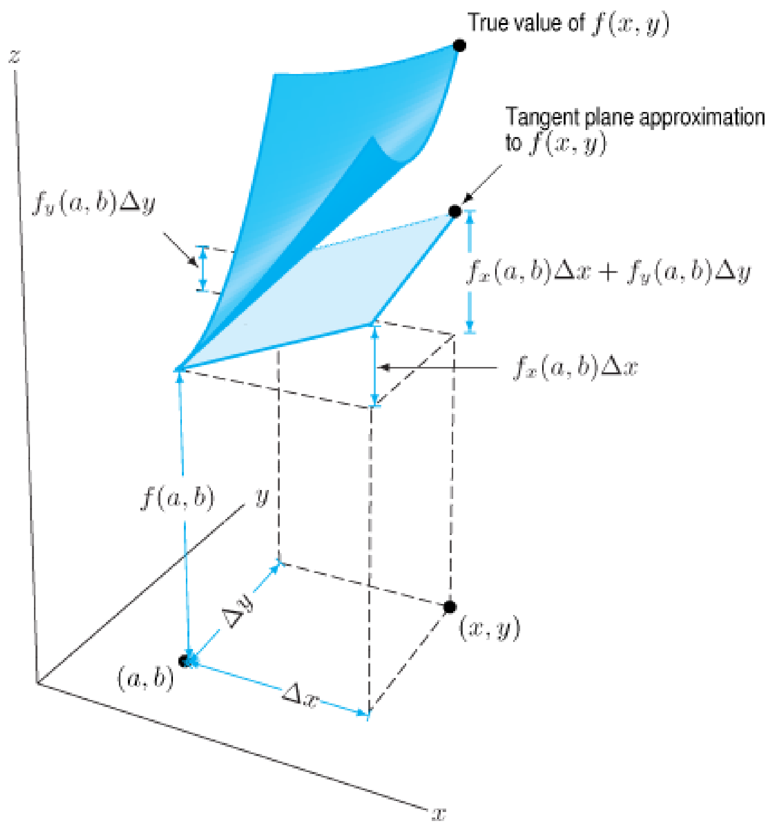
Local Linearization

The idea behind local linearization is that values of the function $f(x, y)$ can be well approximated by z-values of the tangent line to $f(x, y)$ for points close to (a, b).

TANGENT LINE APPROXIMATION TO $f(x, y)$ FOR (x, y) NEAR THE POINT (a, b): Provided f is differentiable at (a, b) we can approximate $f(x, y)$:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We are thinking of a and b as fixed, so the expression on the right hand side is linear in x and y. The right hand side of this approximation gives the *local linearization* of f near $x = a$, $y = b$.



Examples:

3. Find the local linearization of the function $f(x, y) = x^2y$ at the point $(3, 1)$.

Local Linearity with Three or More Variables

Using local linearity to approximate the values of a three variable function $f(x, y, z)$ near a point (a, b, c) will follow the same pattern as that for a two variable function:

$$f(x, y, z) \approx f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c).$$

The Differential

It will sometimes be to our benefit to consider the change in the value of a function $f(x, y)$ as we move a small distance from the point (a, b) to a nearby point (x, y). Writing $\Delta f = f(x, y) - f(a, b)$, $\Delta x = x - a$, and $\Delta y = y - b$, we have

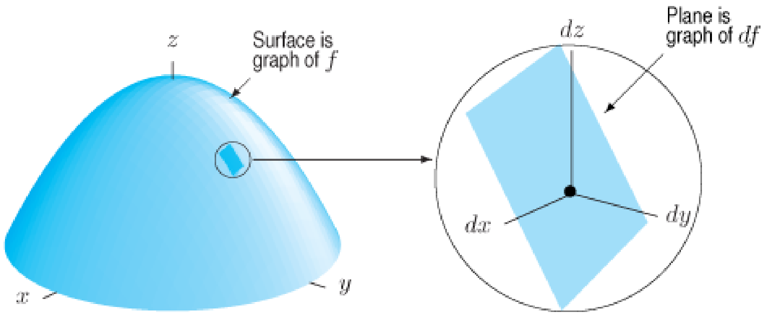
$$\Delta f \approx f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y.$$

Note that for fixed a and b, the right hand side of this approximation is a linear function in Δx and Δy. To define the differential generally, we introduce new variables dx and dy to represent changes in x and y, respectively.

THE DIFFERENTIAL OF A FUNCTION $z = f(x, y)$: The *differential*, df (or dz), at a point (a, b) is the linear function of dx and dy given by the formula

$$df = f_x(a, b)\,dx + f_y(a, b)\,dy$$

The differential at a general point is often written as $df = f_x\,dx + f_y,\,dy$.



Examples:

4. Find the differential of the function:

   (a) $g(u, v) = u^2 + uv$

   (b) $h(x, t) = e^{-3t} \sin(x + 5t)$

   (c) $f(x, y) = xe^{-y}$ at $(1, 0)$.

5. Use your answer from problem 4 part (a) to estimate the change in g as you move from the point $(1, 2)$ to $(1.2, 2.1)$.

6. A right circular cylinder has a radius of 50 cm and a height of 100 cm. Use differentials to estimate the change in the volume of the cylinder if its height and radius are both increased by 1 cm.

7. A student was asked to find the equation of the tangent plane to the surface $z = x^3 - y^2$ at the point $(x, y) = (2, 3)$. The student's answer was

$$z = 3x^2(x - 2) - 2y(y - 3) - 1.$$

At a glance, how do you know that this is wrong? What mistake did the student make? Answer the question correctly.