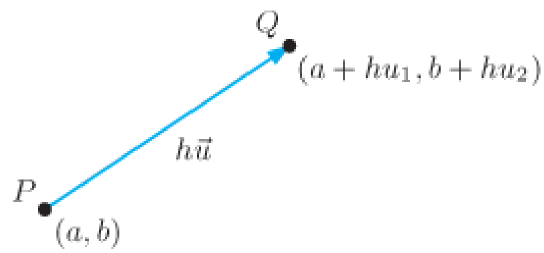
Section 14.4 Gradients and Directional Derivatives in the Plane

So far we are well equipped to find the rate of change of a two-variable function $f(x,y)$ in both the x-direction and the y-direction. But what is so special about those directions? That they point along the coordinate axes? That is special, but it's also a bit arbitrary. It should be clear to all of us that there *is* a rate of change in any given direction. In this section we will outline how to find that rate of change.

Directional Derivative of f at (a,b) in the Direction of the Unit Vector $\vec{u}$

To figure out what the formula for the directional derivative should be, let's assume that we are starting at the point $P=(a,b)$ and we are traveling in the direction of the unit vector $\vec{u}=u_1\vec{i}+u_2\vec{j}$.
First, imagine that $h>0$ is very small, and suppose that we travel from P to the point $Q=(a+hu_1,b+hu_2)$.



DIRECTIONAL DERIVATIVE: If $\vec{u}=u_1\vec{i}+u_2\vec{j}$ is a unit vector, we define the directional derivative $f_{\vec{u}}$ by

$$f_{\vec{u}}(a,b)=\lim_{h\to 0}\frac{f(a+hu_1,b+hu_2)-f(a,b)}{h},$$

provided the limit exists.

Exercise: Use local linearity to find a formula for $f_{\vec{u}}(a,b)$ that involves the partial derivatives $f_x(a,b)$ and $f_y(a,b)$.

USING PARTIAL DERIVATIVES TO COMPUTE DIRECTIONAL DERIVATIVES: The directional derivative $f_{\vec{u}}$ in the direction of the unit vector $\vec{u}$ at the point (a,b) can be computed according to the formula

$$f_{\vec{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2.$$

Examples:

1. Find the directional derivative $f_{\vec{u}}(1,2)$ for the function f with $\vec{u} = (3\vec{i} - 4\vec{j})/5$

   (a) $f(x,y) = 3x - 4y$

   (b) $f(x,y) = x^2 - y^2$

The Gradient Vector

Notice that the formula for the directional derivative looks a lot like a dot product of some vector with the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$. It turns out that this specific vector is very very important.

The Gradient Vector of a differentiable function f at the point (a,b) is

$$\text{grad} f(a,b) = \nabla f(a,b) = f_x(a,b)\vec{i} + f_y(a,b)\vec{j}.$$

A remark on notation: We will sometimes omit the point and write the gradient vector as

$$\operatorname{grad} f = \nabla f = f_x\vec{i} + f_y\vec{j} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}.$$

Notice now, that using the notation of the gradient, we can rewrite the formula for the directional derivative as a dot product:

THE DIRECTIONAL DERIVATIVE AND THE GRADIENT: If f is differentiable at (a, b) and $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is a unit vector, then

$$f_{\vec{u}}(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2 = \operatorname{grad} f(a, b) \cdot \vec{u} = \nabla f \cdot \vec{u}.$$

Examples:

2. Find the gradient of the following function

   (a) $z = xe^y$.

   (b) $f(r, h) = \pi r^2 h$

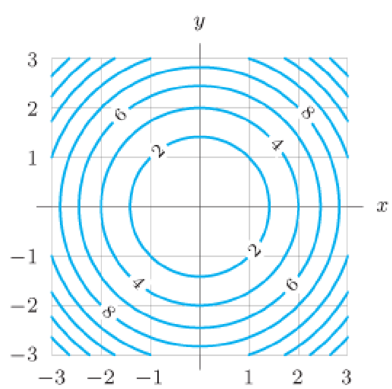
   (c) $f(x, y) = \ln(x^2 + y^2)$

   (d) $z = \arctan(x/y)$.

(e) $z = x\dfrac{e^x}{x+y}$.

3. Find the differential df from the gradient $\nabla f = y\vec{i} + x\vec{j}$.

4. Use the contour diagram below to decide if the directional derivative is positive, negative, or zero.



(a) At the point $(-2, 2)$ in the direction $\vec{i}$.

(b) At the point $(0, -2)$ in the direction $\vec{i} + 2\vec{j}$.