Section 14.5: Gradients and Directional Derivatives in Space

Directional Derivatives of Functions of Three Variables

If $w = f(x, y, z)$ is a function of three variables, and $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, then the derectional derivative of f in the direction of $\vec{u}$ at the point (a, b, c) is given by

$$f_{\vec{u}}(a, b, c) = f_x(a, b, c)u_1 + f_y(a, b, c)u_2 + f_z(a, b, c)u_3.$$

The Gradient Vector of a Function of Three Variables

If $w = f(x, y, z)$ is a function of three variables, the gradient vector, grad f or ∇f, is given by

$$\text{grad } f(a, b, c) = \nabla f(a, b, c) = f_x(a, b, c)\vec{i} + f_y(a, b, c)\vec{j} + f_z(a, b, c)\vec{k}.$$

PROPERTIES OF THE GRADIENT VECTOR IN SPACE: If f is differentiable at (a, b, c) and if $\vec{u}$ is a unit vector, then

$$f_{\vec{u}}(a, b, c) = \nabla f(a, b, c) \cdot \vec{u}.$$

If, in addition, $\nabla f(a, b, c) \neq 0$, then

- $\nabla f(a, b, c)$ is in the direction of the greatest rate of increase of f.
- $\nabla f(a, b, c)$ is perpendicular to the level surface of f passing through (a, b, c).
- $\|\nabla f(a, b, c)\|$ is the maximum rate of change of f at (a, b, c).

TANGENT PLANE TO A LEVEL SURFACE: If $f(x, y, z)$ is differentiable at (a, b, c), then an equation for the tangent plane to the level surface of f passing through the point (a, b, c) is

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0.$$

Examples:

1. Find the directional derivative using $f(x, y, z) = xy + z^2$.

   (a) At the point $(1, 1, 1)$ in the direction of $\vec{i} + 2\vec{j} + 3\vec{k}$.

   (b) At the point $(2, 3, 4)$ in the direction of a vector making an angle of $3\pi/4$ with $\nabla f(2, 3, 4)$.

   (c) At the point $(2, 3, 4)$ in the direction of the maximum rate of change of f.

2. Check that the point $(-1, 1, 2)$ lies on the surface $x^2 - xyz = 3$. Then, viewing the surface as the level surface of a function $f(x, y, z)$, find a vector normal to the surface and an equation for the tangent plane to the surface at $(-1, 1, 2)$.

3. Find an equation for the tangent plane to the surface $3x^2 - 4xy + z^2 = 0$ at the point (a, a, a), where $a \neq 0$.

4. Consider the surface $g(x, y) = 4 - x^2$. What is the relation between $\nabla g(-1, -1)$ and a vector tangent to the path of steepest ascent at $(-1, -1, 3)$? Illustrate your answer with a sketch.

5. Find the equation of the tangent plane to the surface $x^2 + y^2 - xyz = 7$. Do this in two ways.

   (a) Viewing the surface as the level set of a function of three variables $f(x, y, z)$.

   (b) Viewing the surface as the graph of a function of two variables $z = f(x, y)$.

6. Consider the function $f(x, y) = (e^x - x)\cos y$. Suppose S is the surface $z = f(x, y)$.

   (a) Find a vector which is perpendicular to the level curve of f through the point $(2, 3)$ in the direction in which f decreases most rapidly.

   (b) Suppose $\vec{v} = 5\vec{i} + 4\vec{j} - a\vec{k}$ is a vector in $\mathbb{R}^3$ which is tangent to the surface S at the point P lying on the surface above $(2, 3)$. What is a?