Section 14.6: The Chain Rule

The chain rule allows us to differentiate function compositions. We will focus on two cases in particular: the case where $z = f(x, y)$ and $x = g(t)$, $y = h(t)$, and the case where $z = f(x, y)$ while $x = g(u, v)$ and $y = h(u, v)$. From these two main cases, more cases can be extrapolated.

The Chain Rule for $z = f(x, y)$, $x = g(t)$, $y = h(t)$:

First, let's consider what a small change Δz looks like when generated by a small change Δt. To do this, first use local linearity to write an approximation for Δz:

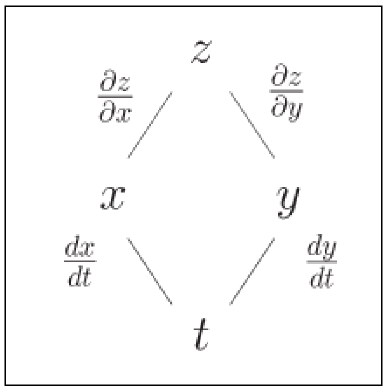
Next, use local linearity for Δx and Δy in terms of Δt to rewrite the expression you came up with above.

Finally, divide everything by Δt and observe what happens to the approximation as $\Delta t \to 0$.

If f, g, and h are differentiable and if $z = f(x, y)$, and $x = g(t)$, and $y = h(t)$, then
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Visualizing the Chain Rule With a Diagram

The diagram below shows a way of remembering the chain rule. I sometimes refer to it as a *dependence tree*. It shows the chain of dependence, in that z depends on x and y, and each x and y depend on t. One must trace z down through both chains to t, multiplying by the correct derivative at each step, to arrive at the chain rule.



Examples:

1. Find dz/dt using the chain rule if $z = \ln(x^2 + y^2)$, $x = t^2$, $y = \ln t$

How to Formulate a General Chain Rule

To find the rate of change of one variable with respect to another variable in a chain of composed differentiable functions:

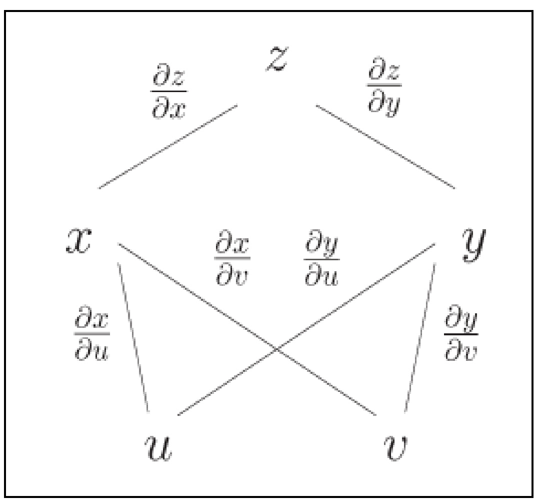
- Draw a diagram expressing the relationship between the variables, and label each link in the diagram with the derivative relating the variables at its ends.
- For each path between two variables, multiply together the derivatives from each step along the path.
- Add the contributions from each path.

For example, if we apply these rules to differentiable functions $z = f(x, y)$ with $x = g(u, v)$ and $y = h(u, v)$, we arrive at the following results:

If f, g, and h are differentiable functions and if $z = f(x, y)$, $x = g(u, v)$, and $y = h(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u},$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Note that the above formulas can be deduced using local linearity in exactly the same way that the formula was deduced before.



Examples:

2. Find $\partial z/\partial u$ and $\partial z/\partial v$ if $z = \ln(xy)$, $x = (u^2 + v^2)^2$, $y = (u^3 + v^3)^2$

3. A bison is charging across the plain one morning. His path takes him to location (x, y) at time t where x and y are functions of t and north is the direction of increasing y. The temperature is always colder farther north. As time passes, the sun rises in the sky, sending out more heat, and a cold front blows in from the east. At time t the temperature H near the bison is given by $f(x, y, t)$. The chain rule expresses the derivative of H with respect to t as the sum of three terms:
$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial t}$$

   Identify the term that gives the contribution to the change in temperature experienced by the bison due to

   (a) The rising sun.

   (b) The coming cold front.

   (c) The bison's change in altitude.