Section 14.7: Second-Order Partial Derivatives

THE SECOND-ORDER PARTIAL DERIVATIVES OF $z = f(x, y)$:

$$\frac{\partial^2 z}{\partial x^2} = f_{xx} = (f_x)_x, \qquad \frac{\partial^2 z}{\partial x \partial y} = f_{yx} = (f_y)_x,$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = (f_x)_y, \qquad \frac{\partial^2 z}{\partial y^2} = f_{yy} = (f_y)_y.$$

Note that the notation from the mixed partials can be thought of in terms of the partial derivative *operators*. For example, we have

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

Examples:

1. Calculate all four second-order partial derivatives. Assume that the variables are restricted to a domain on which the function is defined.

   (a) $f(x, y) = 3x^2y + 5xy^3$

   (b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $f(x, y) = \sin(x^2 + y^2)$

One will notice that in each of the examples above, the mixed partials are equal. That is, $f_{xy} = f_{yx}$. This turns out to be a general result:

THEOREM 14.1: EQUALITY OF MIXED PARTIAL DERIVATIVES: If f_{xy} and f_{yx} are continuous at (a, b), an interior point of their domain, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Taylor Approximations

TAYLOR POLYNOMIAL OF DEGREE 1 APPROXIMATING $f(x, y)$ FOR (x, y) NEAR $(0, 0)$:
If f has continuous first-order partial derivatives, then

$$f(x, y) \approx L(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y.$$

TAYLOR POLYNOMAIL OF DEGREE 2 APPROXIMATING $f(x, y)$ FOR (x, y) NEAR $(0, 0)$:
If f has continuous second-order partial derivatives, then

$$\begin{aligned} f(x, y) &\approx Q(x, y) \\ &= f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{f_{xx}(0, 0)}{2}x^2 + f_{xy}(0, 0)xy + \frac{f_{yy}(0, 0)}{2}y^2. \end{aligned}$$

Examples:

2. Find the quadratic Taylor polynomials about $(0, 0)$ for the following functions

   (a) $f(x, y) = (y - 1)(x + 1)^2$

   (b) $f(x, y) = \dfrac{1}{1 + 2x - y}$

TAYLOR POLYNOMIAL OF DEGREE 1 APPROXIMATING $f(x,y)$ FOR (x,y) NEAR (a,b).
If f has continuous first-order partial derivatives, then

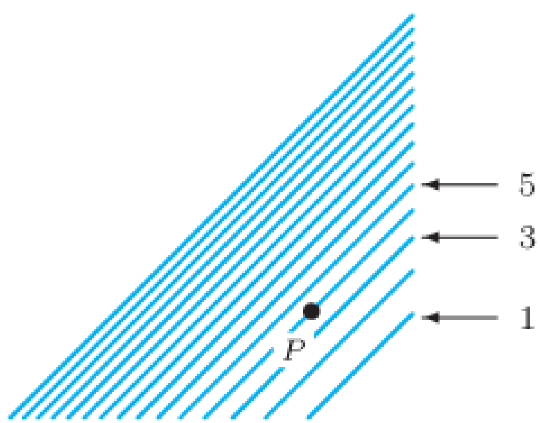
$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

TAYLOR POLYNOMIAL OF DEGREE 2 APPROXIMATING $f(x,y)$ FOR (x,y) NEAR (a,b).
If f has continuous second-order partial derivatives, then

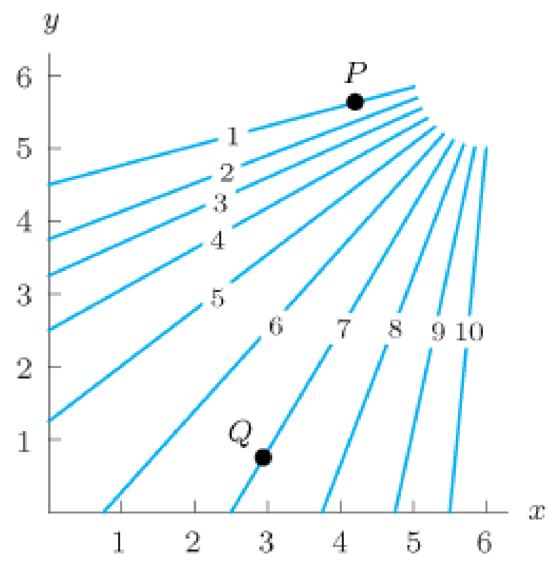
$$f(x,y) \approx Q(x,y)$$
$$= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2.$$

Examples:

3. Use the level curves of the function $z = f(x,y)$ to decide the sign (positive, negative, or zero) of each of $f_x(P)$, $f_y(P)$, $f_{xx}(P)$, $f_{yy}(P)$, and $f_{xy}(P)$.



4. A contour diagram for the smooth function $z = f(x, y)$ is shown in the figure below.



(a) Is z an increasing or decreasing function of x? Of y?

(b) Is f_x positive or negative? How about f_y?

(c) Is f_{xx} positive or negative? How about f_{yy}?

(d) Sketch the direction of ∇f at points P and Q.

(e) Is ∇f longer at point P or at point Q? How do you know?