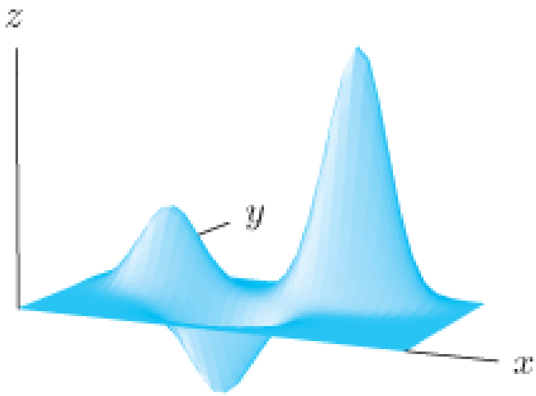
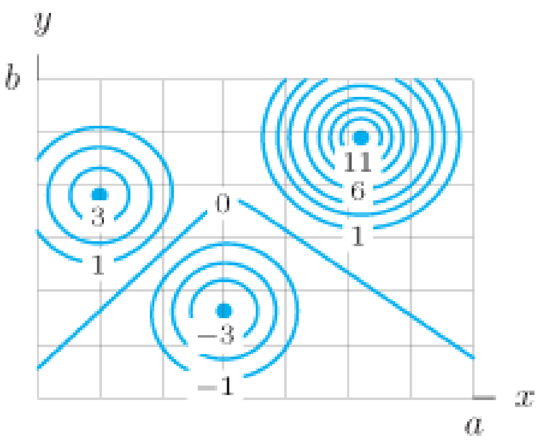
Section 15.1: Critical Points: Local Extrema and Saddle Points

Functions of several variables have local and global extrema in the same way that functions of a single variable do. The definition of a local extrema is more or less the same: a point on the graph of a function where the function reaches a maximum or minimum in some neighborhood around that point. Consider the following graphs:





Here are the quasi-rigorous definitions of local maxima and minima.

- f has a *local maximum* at P_0 if $f(P_0) > f(P)$ for all points P near P_0.
- f has a *local minimum* at P_0 if $f(P_0) < f(P)$ for all points P near P_0.

How do we detect local extrema?

Imagine you are standing on a local maximum of the graph of a two-variable function $z = f(x, y)$. This means that in every direction you travel, you will lose elevation. Recall that ∇f points in the direction of greatest increase. If ∇f were nonzero, this would imply that there is some direction in which you could travel and the function would increase. Since there is no such direction, we must have $\nabla f = \vec{0}$ at the local maximum, so long as ∇f exists at the local maximum. This is how we will define *critical points* of a two-variable function.

DEFINITION: A point P in the domain of a function f is a *critical point* of f if either $\nabla f(P) = \vec{0}$ or $\nabla f(P)$ is undefined.

Finding and Analyzing Critical Points

As you have probably already ascertained, finding points where the gradient vector is the zero vector amounts to setting all of the partial derivatives equal to zero and finding where any of them are undefined.

Examples:

1. Find the critical points of $f(x, y) = x^2 - 2xy + 3y^2 - 8y$ and analyze them.

The Shape of the Graph $f(x, y) = ax^2 + bxy + cy^2$

Suppose a, b, and c are nonzero constants. The function $f(x, y) = ax^2 + bxy + cy^2$ has a critical point at $(0, 0)$ (convince yourself of this). How might we analyze this critical point? We complete the square.

Exercise: Complete the square of $f(x, y) = ax^2 + bxy + cy^2$

We end up with the following result:

$$f(x, y) = a\left[\left(x^2 + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

Looking at it like this, we seee that the shape of this graph is either going to be some elliptical paraboloid, or a saddle shape. The shape will depend on the sign of a, as well as the sign of the coefficient in front of y^2, which is determined by the *discriminant* $D = 4ac - b^2$.

- If $D > 0$, then the expression inside of the square brackets is positive or zero, so the function has a local maximum or minimum.
- If $a > 0$, the function has a local minimum, since the graph is a right-side-up paraboloid.
- If $a < 0$, the function has a local maximum, since the graph is an upside-down paraboloid.
- If $D < 0$, then the function goes up in some directions and down in others (it has a saddle shape). We say the function has a *saddle point*.
- If $D = 0$, then the graph is a parabolid cylinder.

Classifying the Critical Points of a Function

We can use the results from the previous page to help guide us towards how we should classify the critical points of any differentiable function $f(x, y)$. We will begin by considering the case where the function has a critical point at $(0, 0)$ and satisfies $f(0, 0) = 0$.

If $f(0, 0) = 0$, and $\nabla f(0, 0) = \vec{0}$, then the quadratic approximation of $f(x, y)$ is given by

$$f(x, y) \approx Q(x, y) = \frac{f_{xx}(0, 0)}{2}x^2 + f_{xy}(0, 0)xy + \frac{f_{yy}(0, 0)}{2}y^2.$$

The discriminant of $Q(x, y)$ is

$$D = 4ac - b^2 = f_{xx}(0, 0)f_{yy}(0, 0) - (f_{xy}(0, 0))^2$$

If we apply the same logic to this discriminant, we arrive at the *second-derivative test for functions of two variables.*

SECOND DERIVATIVE TEST FOR FUNCTIONS OF TWO VARIABLES: Suppose $\nabla f(x_0, y_0) = \vec{0}$. Let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2.$$

- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then

Examples:

1. Find the critical points and classify them as local maxima, local minima, saddle points, or none of these.

   (a) $f(x, y) = x^2 - 2xy + 3y^2 - 8y$

   (b) $f(x, y) = 2x^3 - 3x^2y + 6x^2 - 6y^2$

(c) $f(x,y) = 8xy - \frac{1}{4}(x+y)^4$

(d) $f(x,y) = e^{2x^2+y^2}$

2. Find A and B so that $f(x, y) = x^2 + Ax + y^2 + B$ has a local minimum of 20 at $(1, 0)$.

3. Which of the points in the figure below appear to be critical points? Classify those that are critical points.