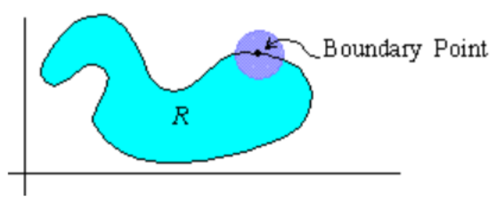
Section 16.1: The Definite Integral of a Function of Two Variables

Recall the definition of the definite integral of a continuous single-variable function f on a closed interval $[a, b]$:

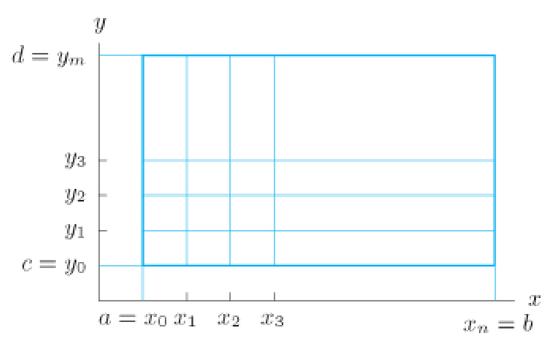
$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_i f(x_i)\,\Delta x.$$

Keep in mind that the region of integration for one variable functions are one-dimensional sets along the real number line (the x-axis). This is because you are integrating over x-values alone. Typically the regions of integration are simply closed intervals.

For a two-variable function $z = f(x, y)$, the regions of integration will then be two-dimensional closed sets in the xy-plane, since we are integrating over points $(x, y) \in \mathbb{R}^2$.



We will begin by considering the simplest type of region in the xy-plane (simplest when we are integrating using Cartesian coordinates): a rectangle of the form $a \le x \le b$, $c \le y \le d$, where a, b, c, d are arbitrary constants:



We now subdivide the region into n subintervals along the x-axis, and m subintervals along the y-axis, effectively subdividing the region into mn subrectangles, each with area $\Delta A = \Delta x\,\Delta y$, where $\Delta x = (b - a)/n$ and $\Delta y = (d - c)/m$.

Notice now that we can choose a random point (u_{ij}, v_{ij}) in the ij-th subrectangle, and form the Riemann sum

$$\sum_{i,j} f(u_{ij}, v_{ij})\,\Delta x\,\Delta y.$$

If $L_{ij} = \min\{f(u_{ij}, v_{ij})\}$ and $M_{ij} = \max\{f(u_{ij}, v_{ij})\}$, then the Riemann sum satisfies

$$\sum_{i,j} L_{ij}\,\Delta x\,\Delta y \le \sum_{i,j} f(u_{ij}, v_{ij})\,\Delta x\,\Delta y \le \sum_{i,j} M_{ij}\,\Delta x\,\Delta y.$$

As Δx, $\Delta y \to 0+$, the lower bound and the upper bound of the Riemann sum converge to the same value, and we arrive at the *double integral*:

Suppose the function f is continuous on R, the rectangle $a \le x \le b$, $c \le y \le d$. If (u_{ij}, v_{ij}) is any point in the ij-th subrectangle, we define the **definite integral** of f over R as

$$\int_R f\,dA = \lim_{\Delta x, \Delta y \to 0+} \sum_{i,j} f(u_{ij}, v_{ij})\,\Delta x\,\Delta y.$$

Such an integral is called a **double integral**.

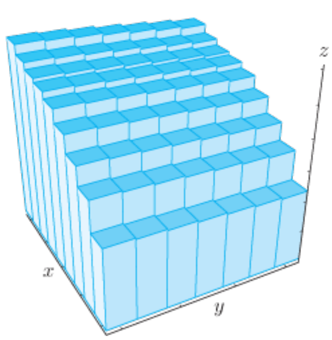
One can also consider regions R in $\mathbb{R}^2$ that are not rectangular. We will briefly describe such regions later. We often think of the *area element* dA as the area of an infinitesimal rectangle of length dx and height dy, so that we can write $dA = dx\,dy$. Then we might use the notation

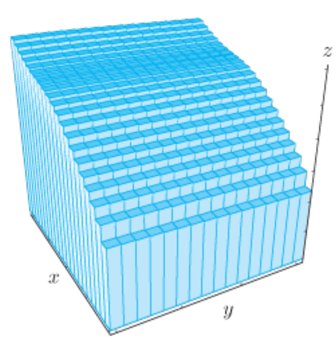
$$\int_R f\,dA = \int_R f(x,y)\,dx\,dy.$$

Interpretation of the Triple Integral as Volume

Recall that for the definite integral of a postive, one-variable function, $\int_a^b f(x)\,dx$, we interpret the Riemann sum as the sum of areas of the form $f(x)\,\Delta x$, and that the sum of the areas converges to the area under the curve and above the x-axis over the interval $[a, b]$.

For the double integral of a positive, two-variable function $f(x, y)$ over a region R, we have a similar interpretation of the double integral as the volume above the region R, and below the surface defined by $z = f(x, y)$.





We therefore arrive at the following result:

> If x, y, and z represent length, and if f is positive, then the volume under the graph of f and above the region R is given by $\displaystyle\int_R f\,dA$.

As with the case with functions of a single-variable, we can make a similar interpretation of the double integral for functions which are sometimes positive, and sometimes negative. If a function $f(x,y)$ goes below the xy-plane on the region R, we interpret the integral as representing the *signed volume*: postive when the surface lies above the xy-plane, and negative when the surface lies below the xy-plane.

Interpretation of the Double Integral as Area

In the special case where $f(x,y) = 1$ for all points (x,y) in a given region R, each term in the Riemann sum is of the form $1 \cdot \Delta A$, and adding up every term of ΔA gives an approximation for the area of the region R. In the limit, this approximation becomes exact, and we have

$$\boxed{\text{Area}(R) = \int_R 1\,dA = \int_R dA.}$$

It might help to imagine the integral as adding up every piece of infinitesimal area dA to accumulate the total area of the region R.

Interpretation of the Double Integral as Average Value

The definite integral can be used, as is the case with one-variable calculus, to compute the average value of a function:

> The average value of f on the region R is given by $\displaystyle\frac{1}{\text{Area}(R)}\int_R f\,dA$.

Integral Over Regions that are not Rectangular

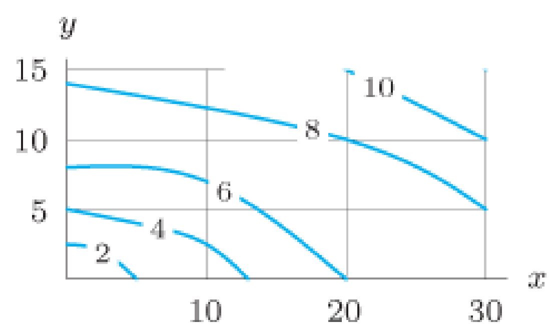
Even though the subdivisions that we used to come up with the definition of the double integral, small pieces of area with $\Delta A = \Delta x\,\Delta y$, are rectangular, it is still possible to define the double integral on regions that are not rectangular. The idea is to use a grid of rectangles that approximates the region. Then as $\Delta A \to 0$, the approximation of the region becomes exact.

Examples:

1. Values of $f(x, y)$ are shown in the table below. Let R be the rectangle $1 \leq x \leq 1.2$, $2 \leq y \leq 2.4$. Find Riemann sums which are reasonable over and underestimates for $\int_R f(x, y)\, dA$ with $\Delta x = 0.1$ and $\Delta y = 0.2$.

y \ x	1.0	1.1	1.2
2.0	5	7	10
2.2	4	6	8
2.4	3	5	4

2. The figure below shows contours of $f(x, y)$ on the rectangle R with $0 \leq x \leq 30$ and $0 \leq y \leq 15$. Using $\Delta x = 10$ and $\Delta y = 5$, find an overestimate and an underestimate for $\int_R f(x, y)\, dA$



3. In the following problems, decide (without any calculation) whether the integrals are positive, negative, or zero. Let D be the region inside the unit circle centered at the origin, Let R be the right half of D, and let B be the bottom half of D.

   (a) $\displaystyle\int_D dA$

   (b) $\displaystyle\int_B 5x\,dA$

   (c) $\displaystyle\int_B (y^3 + y^5)\,dA$

   (d) $\displaystyle\int_B (y - y^3)\,dA$