Section 16.2: Iterated Integrals

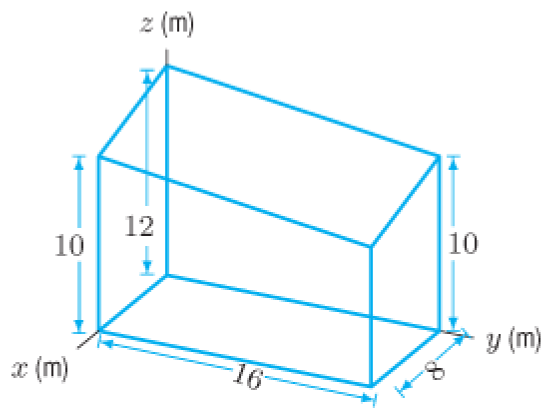
THEOREM 16.1: WRITING A DOUBLE INTEGRAL AS AN ITERATED INTEGRAL: If R is the rectangle $a \leq x \leq b$ and $c \leq y \leq d$ and f is a continuous function on R, then the integral of f over R exists and is equal to the *iterated integral*

$$\int_R f\, dA = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\, dx \right) dy.$$

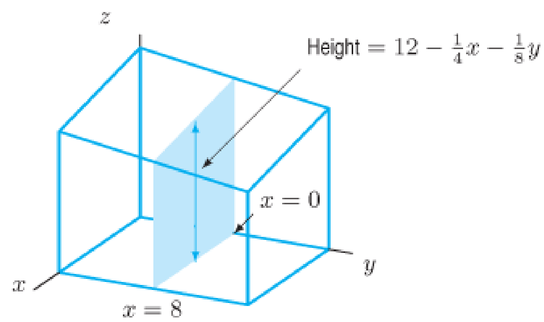
The expression $\int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\, dx \right) dy$ can be written as $\int_c^d \int_a^b f(x,y)\, dx\, dy$.

To evaluate the iterated integral, first perform the inside integral with respect to x, holding y constant; then integrate the result with respect to y.

Example: A building is 8 meters wide and 16 meters long. It has a flat roof that is 12 meters high at one corner, and 10 meters high at each of the adjacent corners. What is the volume of the building?



Notice how the iterated integral is in a way summing over the areas of cross sections:



The Order of Integration:

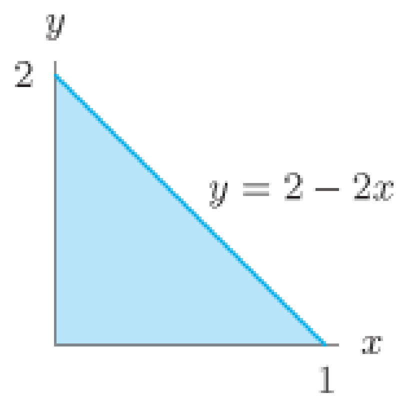
It turns out that with just about every function we are likely to encounter, we can reverse the order of integration. In other words, we have

$$\boxed{\int_R f\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx}$$

Iterated Integrals Over Non-Rectangular Regions

So how does one deal with the situation where the region of integration is not a rectangle? Hopefully the following example helps clarify some issues with regards to that particular nuance.

Example: The density at a point (x, y) of the triangular plate, as shown in the figure below, is $\delta(x, y)$. Express the mass of the plate as an interated integral.



LIMITS ON ITERATED INTEGRALS:

- The limits on the outer integral must be constants.
- The limits on the inner integral can involve only the variable in the outer integral. For example, if the inner integral is with respect to x, its limits can be functions of y.

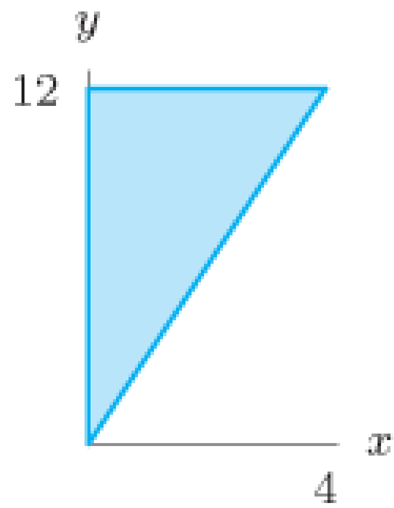
Examples:

1. In the following exercises, sketch the region of integration.

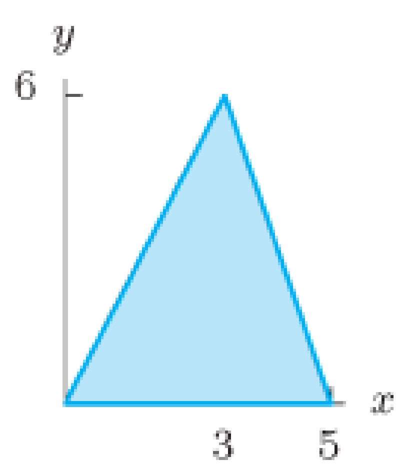
   (a) $\displaystyle\int_0^1 \int_{y^2}^{y} xy\, dx\, dy$

   (b) $\displaystyle\int_0^1 \int_{x-2}^{\cos(\pi x)} y\, dy\, dx$

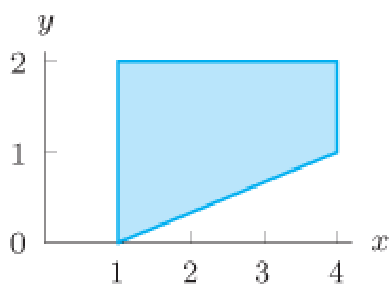
2. In the following exercises, write $\int_R f\, dA$ as an iterated integral for the shaded region R.



(a)



(b)



(c)

3. Evaluate the integral.

(a) $\displaystyle\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$

(b) $\displaystyle\int_0^1 \int_{e^y}^e \frac{x}{\ln x}\, dx\, dy$

4. Set up, but do not evaluate, an iterated integral for the volume of the solid under the graph of $f(x, y) = 25 - x^2 - y^2$ and above the xy-plane.