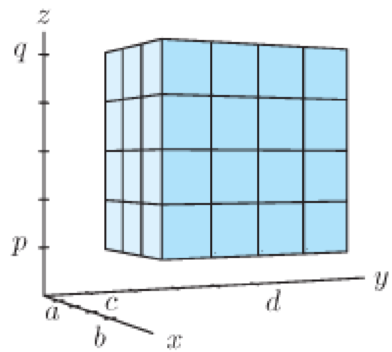
Section 16.3: Triple Integrals

A function of two-variables is integrated over a two-dimensional region $\mathbb{R}^2$. A function of three variables, then, will be integrated over a three-dimensional solid in $\mathbb{R}^3$. We will begin by considering the case where we are integrating a function $f(x, y, z)$ over a rectangular prism W.

We first slice the box up into subdivisions with volume $\Delta V = \Delta x\, \Delta y\, \Delta z$



We pick a point $(u_{ijk}, v_{ijk}, w_{ijk})$ in the ijk-th box, and we form the sum

$$\sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk})\, \Delta V.$$

Then, much as before, we take the limit as Δx, Δy and $\Delta z \to 0$. If f is continuous, the sum converges to the *triple integral* of f over W:

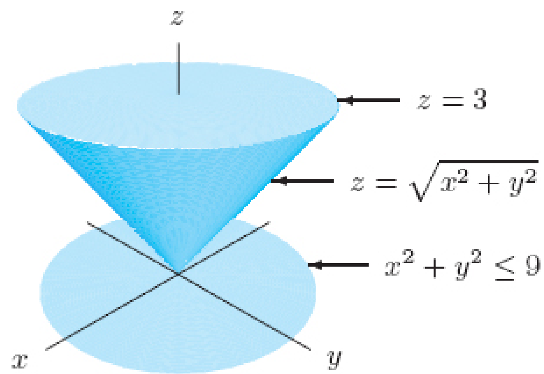
$$\int_W f\, dV = \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk})\, \Delta x\, \Delta y\, \Delta z.$$

TRIPLE INTEGRAL AS AN ITERATED INTEGRAL :

$$\int_W f\, dV = \int_p^q \left(\int_c^d \left(\int_a^b f(x, y, z)\, dx \right) dy \right) dz,$$

where y and z are treated as constants in the innermost (dx) integral, and z is treated as a constant in the middle (dy) integral. Other orders of integration are possible.

Example: Set up an iterated integral to compute the mass of the solid cone bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$, if the density is given by $\delta(x, y, z) = 3$.



LIMITS ON TRIPLE INTEGRALS:

- The limits for the outer integral are constants.
- The limits for the middle integral can involve only one variable (the one in the outer integral).
- The limits for the inner integral can involve two variables (those on the two outer integrals).

Examples:

1. Sketch the region of integration in the following examples.

   (a) $\displaystyle\int_0^1 \int_{-1}^1 \int_0^{\sqrt{1-z^2}} f(x,y,z)\,dy\,dz\,dx$

   (b) $\displaystyle\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_0^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz$

2. Decide whether the integrals are positive, negative, or zero. Let S be the solid sphere $x^2 + y^2 + z^2 \leq 1$, and T be the top half of the sphere (with $z \geq 0$), and B be the bottom half (with $z \leq 0$), and R be the right half (with $x \geq 0$), and L be the left half (with $x \leq 0$).

   (a) $\displaystyle\int_T e^z \, dV$.

   (b) $\displaystyle\int_S \sin z \, dV$.

   (c) $\displaystyle\int_R \sin z \, dV$.

3. Find the volume in the region in the first octant bounded by the coordinate planes and the surface $x + y + z = 2$.

4. Write a triple integral, including limits of integration, for the volume of the solid between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 4$ and above the disk $x^2 + y^2 \leq 1$.