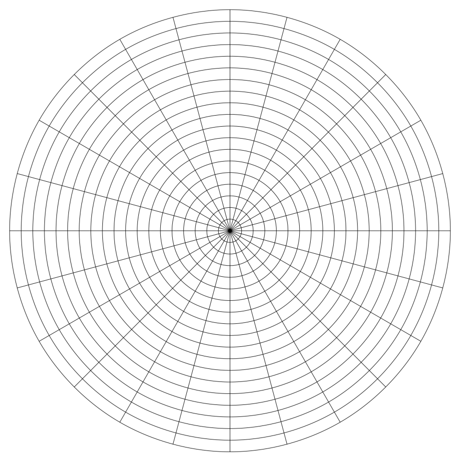
Section 16.4: Double Integrals in Polar Coordinates

Integration in Polar Coordinates

It is often convenient to view $\mathbb{R}^2$ as a *polar grid* instead of a rectangular grid when setting up and computing double integrals. In this case the relationship between the Cartesian coordinates (x, y) and the polar coordinates (r, θ) is given by

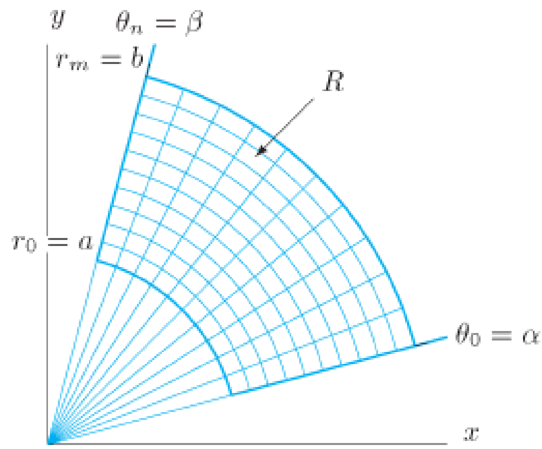
$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ x^2 + y^2 &= r^2. \end{aligned}$$



Now, in order to be able to utilize this coordinate system to integrate effectively, it will be important to determine what form the area element $dA = dx\,dy$ takes when written in polar coordinates.

What is dA in Polar Coordinates?

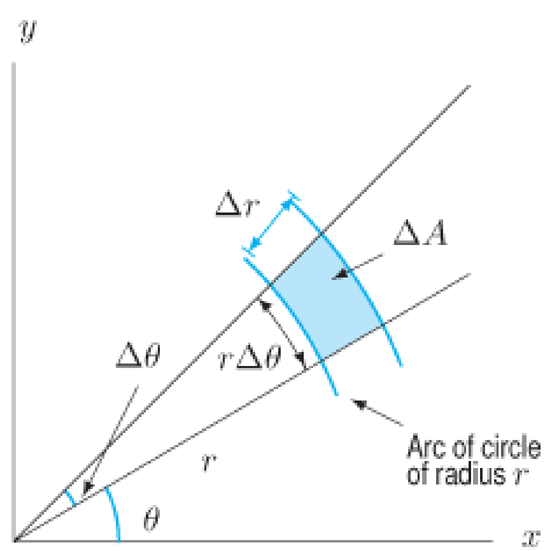
We first note that $r = \text{const}$ gives a circle of constant radius in polar coordinates, while $\theta = \text{const}$ gives a ray emanating from the origing and making an angle of θ with the positive x-axis. A polar grid (such as the one shown above) is built out of such circles and rays. Now, suppose we want to integrate a function $f(x, y) = f(r\cos\theta, r\sin\theta)$ over the region R shown below:



It would be natural to form a Riemann sum where we sum over each of the "bent" rectangles depicted in the figure. Such a Riemann sum would look something like

$$\sum_{i,j} f\,\Delta A.$$

In order to determine what ΔA looks like in the Riemann sum, let us look closer at one of the small bent rectangles.



Analyzing the figure on the previous page, we can see that

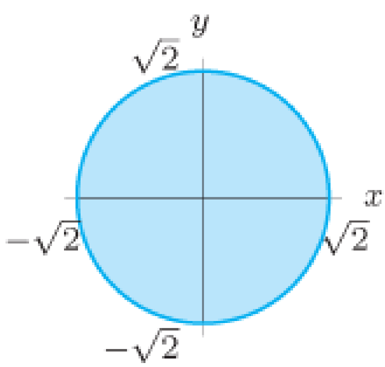
$$\Delta A \approx r \Delta r \, \Delta\theta.$$

Thus we develop the following scheme for integrating in polar coordinates:

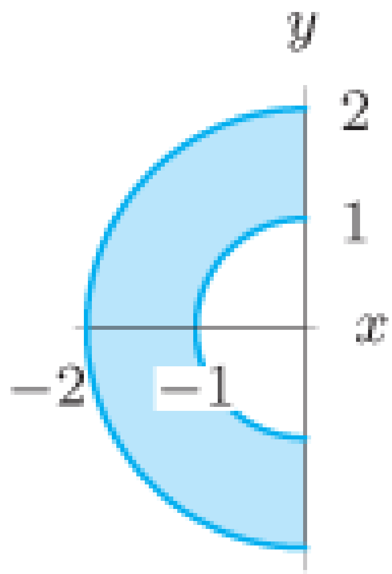
> When computing integrals in polar coordinates, we use $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$. Put $dA = r\,dr\,d\theta$.

Examples:

1. For the following regions R, write $\int_R f\,dA$ as an iterated integral using polar coordinates.



(a)



(b)

2. Sketch the region of integration.

(a) $\int_{\pi/2}^{\pi} \int_{0}^{1} f(r, \theta)\, r\, dr\, d\theta$

(b) $\displaystyle\int_{\pi/6}^{\pi/3} \int_0^1 f(r,\theta)\, r\, dr\, d\theta.$

(c) $\displaystyle\int_0^{\pi/4} \int_0^{1/\cos\theta} f(r,\theta)\, r\, dr\, d\theta.$

(d) $\displaystyle\int_4^3 \int_{3\pi/4}^{3\pi/2} f(r,\theta)\, r\, dr\, d\theta.$

3. Evaluate the integral

(a) $\int_R \sqrt{x^2+y^2}\, dA$, where R is $4 \leq x^2+y^2 \leq 9$.

(b) $\int_0^2 \int_y^{\sqrt{4-y^2}} xy\, dx\, dy$.

4. Sketch the region of integration and evaluate:

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x\, dy\, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x\, dy\, dx$$

5. Find the volume of the region between the graph of $f(x, y) = 25 - x^2 - y^2$ and the xy-plane.