Section 17.1: Parametrized Curves

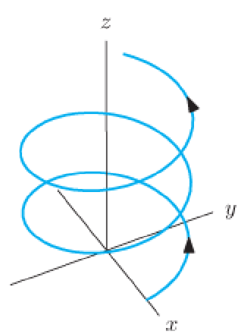
A curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be parametrized by a series of functions of a single variable, one for each of the coordinates. In $\mathbb{R}^3$, this means that a curve can be parametrized by three functions $x = f(t)$, $y = g(t)$, and $z = h(t)$, where t is allowed to vary over some domain on the real numbers. As t varies, the coordinates (x, y, z) trace out a curve in space.

Examples:

1. Find parametric equations for the curve $y = x^3$ in the xy-plane.

2. Describe in words the motion given parametrically by

$$x = \cos t, \qquad y = \sin t, \qquad z = t.$$



3. Write parametric equations for the circle of radius 2, centered at the origin in the xy-plane.

4. Find parametric equations for the line in the direction of the vector $\vec{i} + 2\vec{j} - \vec{k}$ and through the point $(3, 0, -4)$

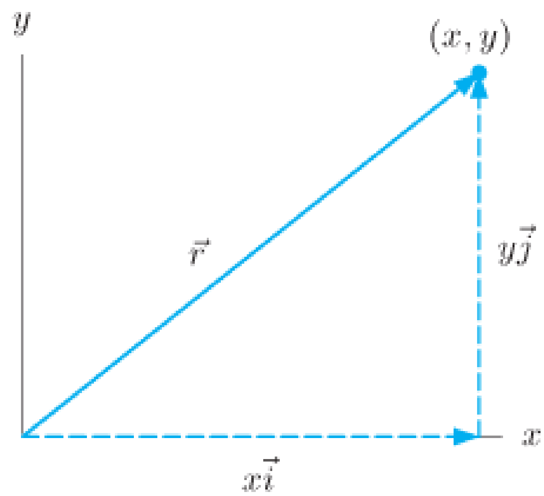
The previous example can be generalized to show how to obtain the parametric equations for a line that passes through any arbitrary point and points in the direction of any vector.

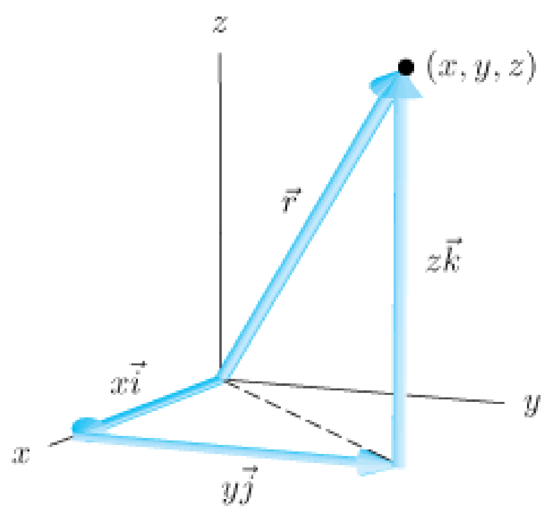
PARAMETRIC EQUATIONS OF A LINE through the point (x_0, y_0, z_0) and parallel to the vector $a\vec{i} + b\vec{j} + c\vec{k}$ are

$$x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct.$$

Using Position Vectors to Write Parametrized Curves as Vector-Valued Functions

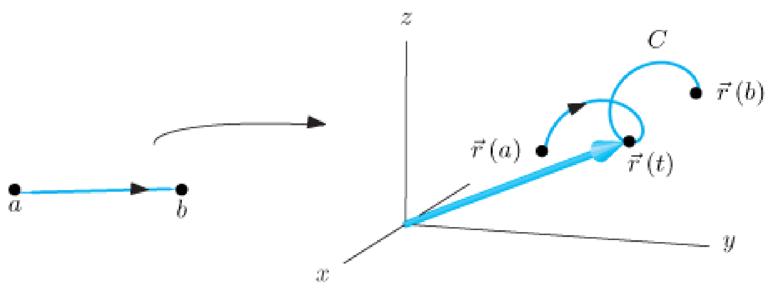
As we noted before, there is a one-to-one correspondence between points in $\mathbb{R}^2$ and $\mathbb{R}^3$ and vectors. Any point (x, y) in $\mathbb{R}^2$ can be represented by the *position vector* $\vec{r} = x\vec{i} + y\vec{j}$. Similarly, any point (x, y, z) in $\mathbb{R}^3$ can be represented by the position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.





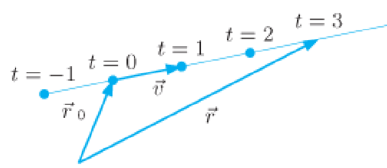
If $x = f(t)$, $y = g(t)$, and $z = h(t)$ are parametric equations for a curve in $\mathbb{R}^3$, we can now combine these three equations into the single *vector equation*

$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}.$$



Parametric Equations of a Line

As before, consider a straight line in the direction of a vector $\vec{v}$ which passes through the point (x_0, y_0, z_0) with position vector $\vec{r}_0$. The idea is to start at $\vec{r}_0$, and move up the line, adding different multiples of $\vec{v}$.



PARAMETRIC EQUATION OF A LINE IN VECTOR FORM: The line through the point with position vector $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$ in the direction of the vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ has parametric equation

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}.$$

Examples:

5. Parametrize the line perpendicular to the plane $z = 2x - 3y + 7$ and through the point $(1, 1, 6)$.

6. Parametrize the circle of radius 2 parallel to the xy-plane, centered at the point $(0, 0, 1)$, and traversed counterclockwise when viewed from below.

7. Parametrize the curve in which the plane $z = 2$ cuts the surface $z = \sqrt{x^2 + y^2}$.

8. Parametrize the line through $P = (2, 5)$ and $Q = (12, 9)$ so that the points P and Q correspond to the parameter values $t = 0$ at P and $t = 5$ at Q.

9. Find an equation for the plane containing the point $(2, 3, 4)$ and the line $x = 1 + 2t$, $y = 3 - t$, and $z = 4 + t$.