Section 17.2: Motion, Velocity, and Acceleration

The Velocity Vector

We covered the notion of the velocity vector back in chapter 13. However, now that we are able to parametrize curves, we will be able to utilize the concept to a greater degree than before.

The *velocity vector* of a moving object is a vector $\vec{v}$ such that:

- The magnitude of $\vec{v}$ is the speed of the object.
- The direction of $\vec{v}$ is the direction of motion.

Thus the speed of the object is $\|\vec{v}\|$ and the velocity vector is tangent to the object's path.

Computing the Velocity

If $\vec{r}(t)$ represents the position of a particle at time t, then the displacement of the particle from t to $t + \Delta t$ is given by $\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$. Over this time interval, we have

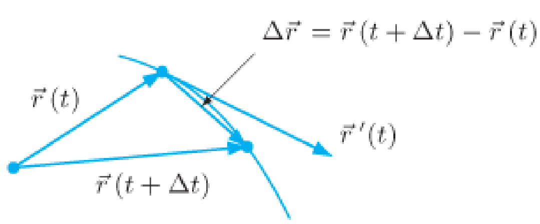
$$\text{Average Velocity} = \frac{\Delta\vec{r}}{\Delta t}.$$

If we take the limit as $\Delta t \to 0$, we get the instantaneous velocity:

The velocity vector, $\vec{v}(t)$, of a moving object with position vector $\vec{r}(t)$ at time t is

$$\vec{v}(t) = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t},$$

whenever the limit exists. We use the notation $\vec{v} = \dfrac{d\vec{r}}{dt} = \vec{r}\,'(t)$.



Exercise: Show that the components of the velocity vector are given by

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

The *components of the velocity vector* of a particle moving in space with position vector $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ at time t are given by

$$\vec{v}(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}.$$

Examples

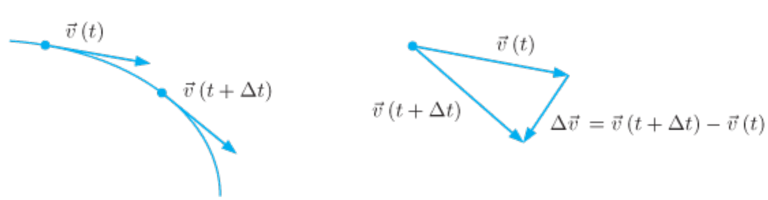
1. Find the velocity $\vec{v}(t)$ and the speed $\|\vec{v}(t)\|$ if the position of a particle is given by $x = \cos(3t)$, $y = \sin(5t)$. Find any times at which the particle stops.

The Acceleration Vector

The *acceleration vector* of a moving object with velocity $\vec{v}(t)$ at time t is

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t},$$

if the limit exists. We use the notation $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2} = \vec{r}''(t)$.



The *components of the acceleration vector*, $\vec{a}(t)$, at time t of a particle moving in space with position vector $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ at time t are given by

$$\vec{a}(t) = f''(t)\vec{i} + g''(t)\vec{j} + h''(t)\vec{k} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}.$$

Examples

2. A child is sitting on a ferris wheel of diameter 10 meters, making one revolution every 2 minutes. Find the speed of the child, find the velocity vector $\vec{v}(t)$, and find the acceleration vector, $\vec{a}(t)$. Draw the velocity vector and the acceleration vector at some instant in time. Assume that the center of the ferris wheel is at the origin, and that the ferris wheel makes its rotations counterclockwise.

UNIFORM CIRCULAR MOTION: For a particle whose motion is described by

$$\vec{r}(t) = R\cos(\omega t)\vec{i} + R\sin(\omega t)\vec{j}$$

- Motion is in a circle of radius R with period $2\pi/|\omega|$.
- Velocity, $\vec{v}$, is tangent to the circle and speed is constant $\|\vec{v}\| = |\omega|R$.
- Acceleration, $\vec{a}$, points toward the center of the circle with $\|\vec{a}\| = \|\vec{v}\|^2/R$.

MOTION IN A STRAIGHT LINE: For a particle whose motion is described by

$$\vec{r}(t) = \vec{r}_0 + f(t)\vec{w}$$

- Motion is along a straight line through the point with position vector $\vec{r}_0$ and parallel to $\vec{w}$.
- Velocity, $\vec{v}$, and acceleration, $\vec{a}$, are parallel to the line.

Examples

3. Find the velocity and acceleration vectors.

   (a) $x = 2 + 3t^2$, $y = 4 + t^2$, $z = 1 - t^2$.

   (b) $x = 3\cos(t^2)$, $y = 3\sin(t^2)$, $z = t^2$.

4. Find all the values of t for which the particle is moving parallel to the x-axis and to the y-axis. Determine the end behavior and graph the particle's path.

The Length of a Curve

We can use the concepts of position and velocity to help us derive a formula for the length of a curve. Recall that the speed of a particle is the magnitude of its velocity vector:

$$\text{Speed} = \|\vec{v}\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

As was the case back in one-variable calculus, we can find the total distance traveled by the particle from $t = a$ to $t = b$ by integrating the speed:

$$\text{Distance traveled} = \int_a^b \|\vec{v}(t)\|\, dt$$

Now, so long as the particle never stops or reverses direction along the curve, the distance the particle travels will be the same as the length of the curve.

If the curve C is given parametrically for $a \leq t \leq b$ by smooth functions and if the velocity vector $\vec{v}$ is not $\vec{0}$ for $a < t < b$, then

$$\text{Length of } C = \int_a^b \|\vec{v}(t)\|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Examples

5. Find the length of the curve given by

$$\vec{r}(t) = 2t\vec{i} + \ln t\vec{j} + t^2\vec{k}$$

for $1 \leq t \leq 2$.

6. A particle moves at a constant speed along a line from the point $P = (2, -1, 5)$ at time $t = 0$ to the point $Q = (5, 3, -1)$. Find parametric equations for the particle's motion if

   (a) The particle takes 5 seconds to move from P to Q.

   (b) The speed of the particle is 5 units per second.

7. Consider the parametric curve $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$.

   (a) Find a parametric equation $\vec{q}(s)$ of the tangent line to the curve at time $t = T$.

   (b) If $T = 2$, find the point of intersection of the tangent line with the xy-plane.