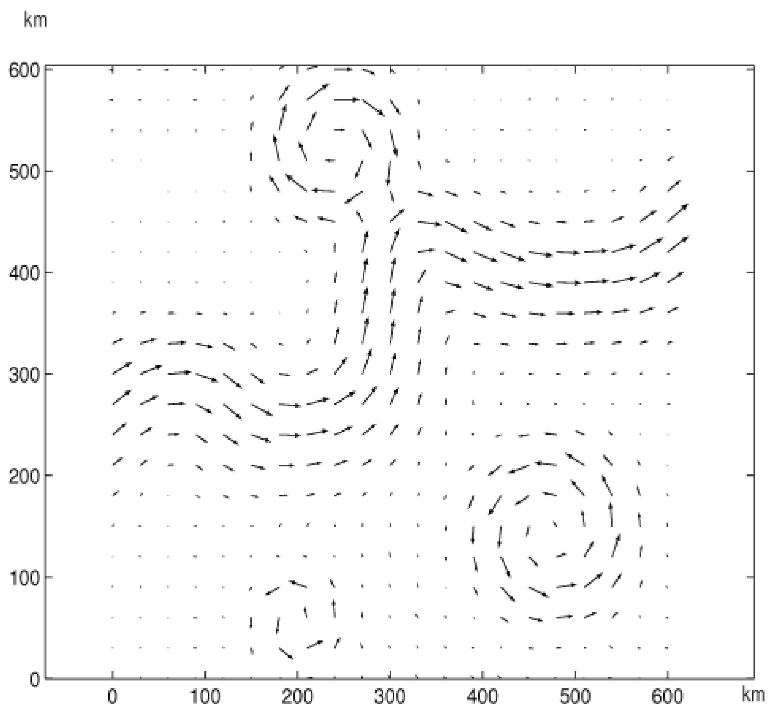
Section 17.3: Vector Fields

A *vector field* is a function which assigns a unique vector to each point in $\mathbb{R}^2$ or $\mathbb{R}^3$. An obvious example of a vector field would be the gradient field of a function $f(x, y)$ or $f(x, y, z)$.



The image above shows the velocity vector field of the gulf stream.

Force Fields

A very important vector quantity that will give rise to many of our vector fields is force. For example, the earth exerts a gravitational force at every point in space. Such a vector field would look like the following:

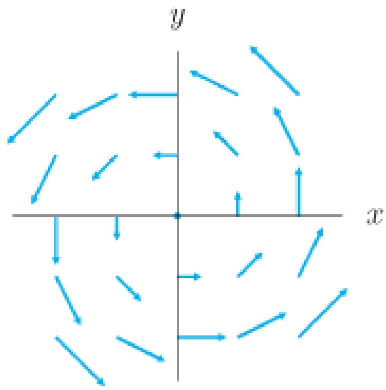
Definition of a Vector Field

A *vector field* in $\mathbb{R}^2$ is a function $\vec{F}(x, y)$ whose value at a point (x, y) is a two-dimensional vector. Similarly, a vector field in $\mathbb{R}^3$ is a function $\vec{F}(x, y, z)$ whose value at a point (x, y, z) is a three-dimensional vector.

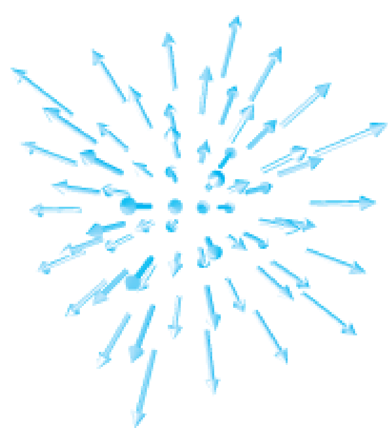
Often we will identify a point (x, y) or a point (x, y, z) with its position vector $\vec{r}$ and write our vector field as $\vec{F}(\vec{r})$.

Examples:

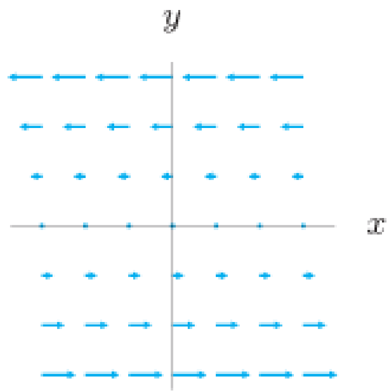
1. Sketch the vector field in $\mathbb{R}^2$ given by $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$.



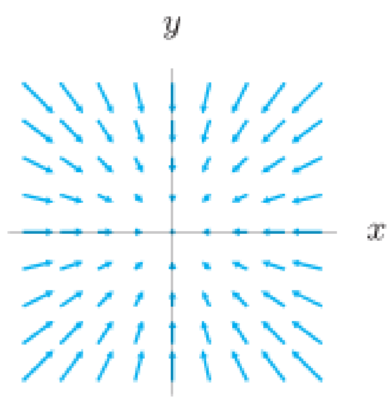
2. Describe the vector field in $\mathbb{R}^3$ given by $\vec{F}(\vec{r}) = \vec{r}$.



3. Find a possible formula for the given vector fields.



(a)



(b)

4. Sketch the vector fields in the xy-plane.

(a) $\vec{F}(x, y) = y\vec{i}$.

(b) $\vec{F}(\vec{r}) = 2\vec{r}$.

(c) $\vec{F}(\vec{r}) = -\dfrac{\vec{r}}{\|\vec{r}\|^3}$

5. In the following, give an example of a vector field $\vec{F}(x, y)$ in $\mathbb{R}^2$ with the stated properties.

   (a) $\vec{F}$ is constant.

   (b) $\|\vec{F}\|$ is constant but $\vec{F}$ is not constant.

   (c) $\vec{F}$ is perpendicular to $\vec{G} = (x + y)\vec{i} + (1 + y^2)\vec{j}$ at every point.

6. Write a formula for a vector field with the given properties.

   (a) All vectors are parallel to the x-axis; all vectors on a vertical line have the same magnitude.

   (b) All vectors are of unit length and perpendicular to the position vector at that point.

7. Newton's Law of Gravitation states that the magnitude of the gravitational force exerted on an object of mass M on an object of mass m is proportional to M and m and inversely proportional to the square of the distance between them. The direction of the force is from m to M along the line connecting them. Find a formula for $\vec{F}(\vec{r})$ that represents the gravitational force, assuming M is located at the origin and m is located at the point with position vector $\vec{r}$.