Section 18.1: The Idea of a Line Integral

In the past we spoke of the concept of rowing in a current, and we spoke of ways in which we might use the dot product to determine how much of the current was working with or against your efforts. We can utilize the same concept to determine how much of any vector is working with or against another vector.

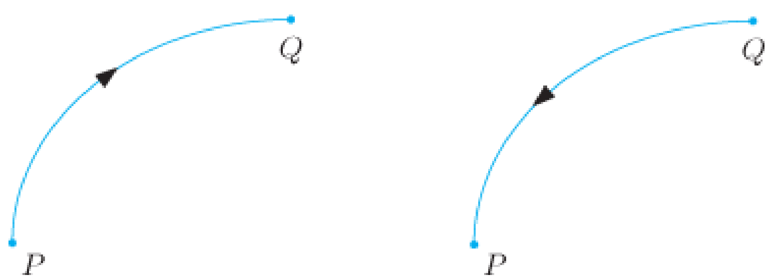
Of course, back then, we were speaking of a situation in which we were dealing with something like a constant velocity vector and a constant displacement vector.

Now, we will utilize the same basic principles to learn how we might develop a method for computing how much of a *vector field* is flowing with or against a curved trajectory in space. In other words, the *line integral*, which we define in this section, measures the extent to which a curve in a vector field is going with or against the vector field overall.

Orientation of a Curve

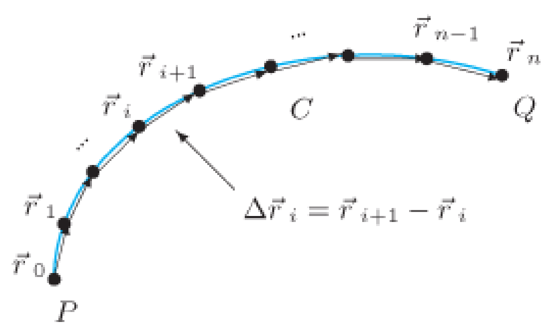
The concept of orientation of a curve is simple enough to understand. A curve can be traced out in one of two directions. Choosing one of those directions determines an *orientation* of the curve.

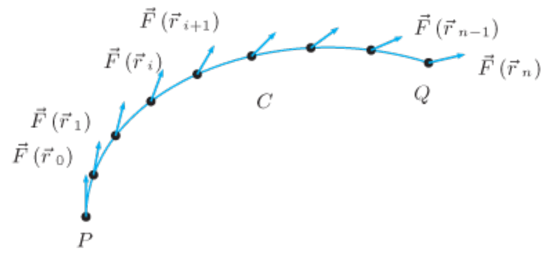
> A curve is said to be *oriented* if we have chosen a direction of travel on it.



Definition of the Line Integral

Suppose $\vec{F}$ is a vector field (either in $\mathbb{R}^2$ or $\mathbb{R}^3$), and C is an oriented curve. We develop the line integral the way that we develop all integrals, in this case by first slicing up the curve C into n small, approximately straight pieces along which $\vec{F}$ is approximately constant. Then there is a displacement vector $\Delta\vec{r}_i = \vec{r}_{i+1} - \vec{r}_i$ associated with each piece, and the value of $\vec{F}$ is approximately $\vec{F}(\vec{r}_i)$.





Now, along each displacement vector $\Delta\vec{r}_i$, we can compute the dot product $\vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$ to measure the extent to which $\vec{F}$ points along the curve or against the curve at $\vec{r}_i$.

Summing the dot products over all such pieces, we arrive at a Riemann sum:

$$\sum_{i=0}^{n-1}\vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

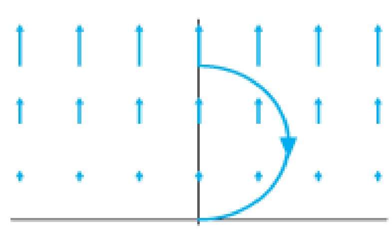
Taking the limit as $\|\vec{r}_i\|\to 0$, we arrive at the definition of the line integral, provided that the limit exists.

The *line integral* of a vector field $\vec{F}$ along an oriented curve C is

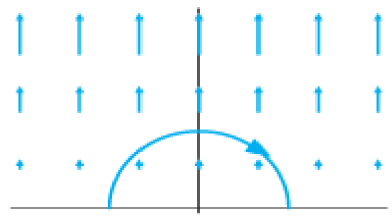
$$\int_C \vec{F}\cdot d\vec{r} = \lim_{\|\vec{r}_i\|\to 0}\sum_{i=0}^{n-1}\vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i.$$

Examples:

1. Say whether you expect the line integral of the pictured vector field over the given curve to be positive, negative, or zero.



(a)



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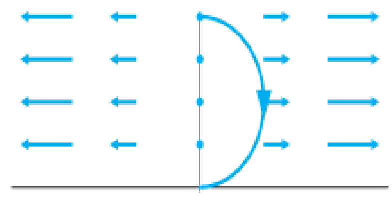


(c)

2. Calculate the line integral of the vector field along the line between the given points.

   (a) $\vec{F} = x\vec{j}$ from $(2, 0)$ to $(2, 5)$.

   (b) $\vec{F} = 6\vec{i}$ from $(2, 0)$ to $(6, 0)$.

   (c) $\vec{F} = 6x\vec{i} + (x + y^2)\vec{j}$, where C is the x-axis from $(0, 3)$ to $(0, 5)$.

Interpretations of the Line Integral: Work

Recall that if a constant force $\vec{F}$ acts on an object over a displacement $\vec{d}$, the work done by the force is given by

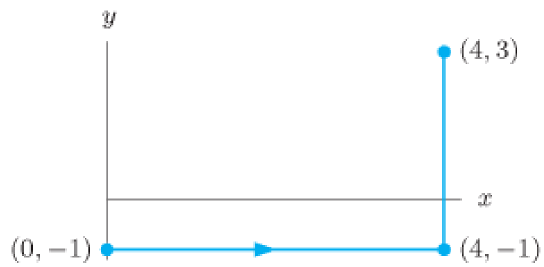
$$\text{Work done} = \vec{F} \cdot \vec{d}$$

Now consider the way that the line integral was developed. If an object is moving along an oriented curve C and a force field $\vec{F}$ is doing work on that object, along each displacement vector $\Delta\vec{r}_i$, the work done by $\vec{F}$ is approximately $\vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$. Summing up the work done over each displacement $\Delta\vec{r}_i$ and taking the limit as $\|\Delta\vec{r}_i\| \to 0$, we arrive at the following result:

$$\boxed{\text{Work Done by } \vec{F} = \int_C \vec{F} \cdot d\vec{r}.}$$

Examples:

3. An object moves along a curve C shown in the figure below while being acted upon by the force field $\vec{F}(x, y) = y\vec{i} + x^2\vec{j}$. Find the work done by $\vec{F}$.



Circulation

If C is an oriented, closed curve, the line integral of a vector field F around C is called the *circulation* of $\vec{F}$ around C.

We will often use the notaton $\oint_C \vec{F} \cdot d\vec{r}$ to refer to circulations.