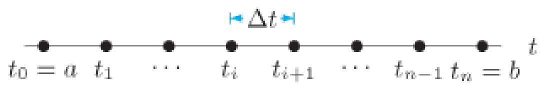
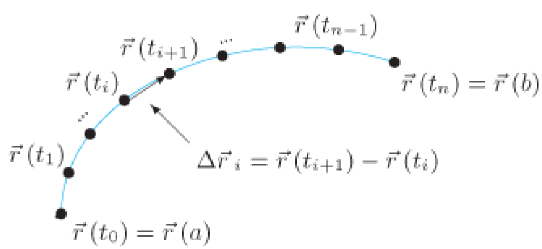
Section 18.2: Computing Line Integrals Over Parametrized Curves

Recall the definition of the line integral:

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\|\vec{r}_i \to 0} \sum \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i$$

Now, suppose we have a smooth parametrization $\vec{r}(t)$ for our curve C for $a \le t \le b$. Then if we divide the interval $a \le t \le b$ into n equal subdivisions, each of length $\Delta t = (b-a)/n$, we automatically get a subdivision of the curve C:





Then, at each point $\vec{r}(t_i)$, we compute the dot product

$$\vec{F} \cdot \Delta\vec{r}_i$$

as follows:

$$\begin{aligned}
\Delta\vec{r}_i &= \vec{r}(t_{i+1}) - \vec{r}(t_i) \\
&= \vec{r}(t_i + \Delta t) - \vec{r}(t_i) \\
&= \frac{\vec{r}(t_i + \Delta t) - \vec{r}(t_i)}{\Delta t} \cdot \Delta t \\
&\approx \vec{r}\,'(t_i)\,\Delta t,
\end{aligned}$$

where the last approximation comes from the fact that Δt is small and the parametrization $\vec{r}(t)$ is smooth.

Using this, we obtain one last approximation for the sum defining the line integral:

$$\sum_{i=0}^{n-1} \vec{F}(\vec{r}_i) \cdot \Delta\vec{r}_i \approx \sum_{i=0}^{n-1} \vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\,\Delta t.$$

The real kicker here is that the quantity $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$ is itself a one-variable function of t. Therefore, the sum on the right is a Riemann sum which will converge to a single definite integral of the form

$$\int_a^b g(t)\, dt,$$

where

$$g(t) = \vec{F}(t) \cdot \vec{r}'(t).$$

Thus, as $\Delta t \to 0$, we obtain the following:

$$\lim_{\Delta t \to 0} \sum \vec{F}(\vec{r}(t_i)) \cdot \vec{r}'(t_i)\, \Delta t = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt.$$

What we have done is developed a formula for computing line integrals over parametrized curves. This formula is extremely handy, and gives us a way of boiling down something as complex as a line integral into the definite integral of a single variable function over a closed interval:

> If $\vec{r}(t)$, for $a \le t \le b$, is a smooth parametrization of the oriented curve C and $\vec{F}$ is a vector field that is continuous on C, then
>
> (1) $$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt.$$
>
> In words: To compute the line integral of $\vec{F}$ over C, take the dot product of $\vec{F}$ evaluated on C with the velocity vector, $\vec{r}'(t)$, of the parametrization C, then integrate along the curve.

Examples:

1. Write $\int_C \vec{F} \cdot d\vec{r}$ in the form $\int_a^b g(t)\, dt$

   (a) $\vec{F} = y\vec{i} + x\vec{j}$ and C is the semicircle from $(0,1)$ to $(0,-1)$ with $x > 0$.

(b) $\vec{F} = x\vec{i} + z^2\vec{k}$ and C is the line from $(0,1,0)$ to $(2,3,2)$.

2. Find the line integral.

   (a) $\int_C (x\vec{i} + y\vec{j}) \cdot d\vec{r}$, where C is the semicircle with center at $(2,0)$ and going from $(3,0)$ to $(1,0)$ in the region $y > 0$.

(b) $\int_C ((x^2+y)\vec{i}+y^3\vec{j})\cdot d\vec{r}$ where C consists of the three line segments from $(4,0,0)$ to $(4,3,0)$ to $(0,3,0)$ to $(0,3,5)$.

3. Find $\int_C \vec{F} \cdot d\vec{r}$ for the given $\vec{F}$ and C.

(a) $\vec{F} = \ln y\,\vec{i} + \ln x\,\vec{j}$ and C is the curve given parametrically by $\langle 2t, t^3 \rangle$, for $2 \le t \le 4$.

(b) $\vec{F} = e^y\vec{i} + \ln(x^2+1)\vec{j} + \vec{k}$ and C is the circle of radius 2 centered at the origin in the yz-plane, shown below: