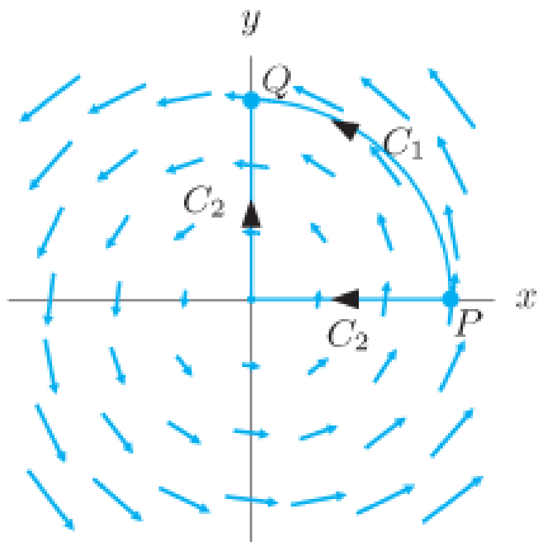
Section 18.4: Path Dependent Vector Fields and Green's Theorem

One obvious way to tell confirm that a vector field is path dependent is to compute a line integral of the vector field along multiple piecewise smooth curves connecting points P and Q. If the value of the line integral changes from one curve to the next, then the vector field is path dependent.



As we can see, the vector field pictured above is path dependent. What can you say about a circulation $\oint_C \vec{F} \cdot d\vec{r}$ in this case?

The following is an immediate result of the FTCLI

> A vector field is conservative if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C.

How To Tell if a Vector Field is Path-Independent Algebraically: The Curl

Consider a two dimensional vector field $\vec{F} = F_1\vec{i} + F_2\vec{j}$. If $\vec{F}$ is conservative, then there is a scalar function f such that

$$\vec{F} = F_1\vec{i} + F_2\vec{j} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j},$$

from which we can conclude that

$$F_1 = \frac{\partial f}{\partial x} \text{ and } F_2 = \frac{\partial f}{\partial y}.$$

If the components of $\vec{F}$ have continuous partial derivatives, we have

$$\frac{\partial F_1}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial F_2}{\partial x}$$

We arrive at the following result:

If $\vec{F}(x,y) = F_1\vec{i} + F_2\vec{j}$ is a vector field with continuous partial derivatives, then

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

If $\vec{F}(x,y) = F_1\vec{i} + F_2\vec{j}$ is an arbitrary vector field, then we define the 2-dimensional scalar *curl* of the vector field $\vec{F}$ to be

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

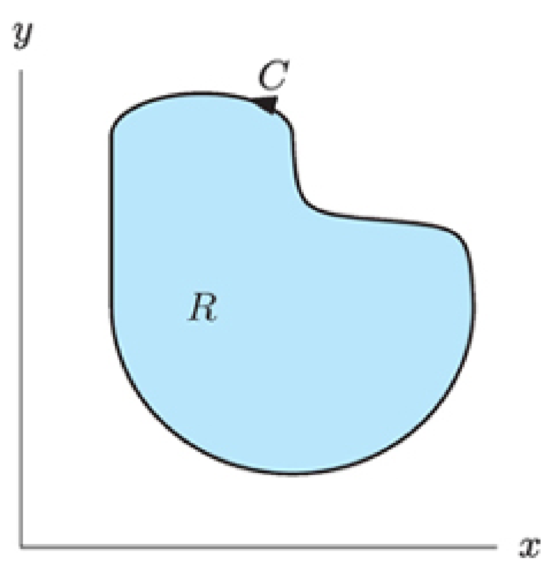
Green's Theorem

We know now that if $\vec{F} = \nabla f$ for some potential function f, then $\partial F_2/\partial x - \partial F_1/\partial y = 0$. A natural question is whether or not the implication works in the other direction. That is, if the scalar curl of a vector field is zero, can we conclude that the vector field is in fact conservative? This is a difficult question to answer.

But given everything we know, we should suspect that there is a relationship between a circulation $\oint_C \vec{F} \cdot d\vec{r}$ and the scalar curl of F. This relationship can be described by *Green's Theorem*.

GREEN'S THEOREM:
Suppose C is a piecewise smooth simple closed curve that is the boundary of a region R in the plane and oriented so that the region is on the left as we move around the curve. Suppose $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a smooth vector field on an open region containing R and C. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy.$$



Examples:

1. Use Green's Theorem to evaluate $\oint_C (y^2\vec{i} + x\vec{j}) \cdot d\vec{r}$ where C is the counterclockwise path around the perimeter of the rectangle $0 \le x \le 2$, $0 \le y \le 3$.

The Curl Test for Vector Fields in the Plane

Assuming the results from Green's Theorem, it is now easy to see that the reverse implication we discussed from above is indeed true. That is,

> THE CURL TEST FOR VECTOR FIELDS IN $\mathbb{R}^2$:
> Suppose $\vec{F} = F_1\vec{i} + F_2\vec{j}$ is a vector field with continuous partial derivatives such that
>
> - The domain of $\vec{F}$ has the property that every closed curve in it encircles a region that lies entirely within the domain. In particular, the domain of $\vec{F}$ has no holes.
> - $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0.$
>
> Then $\vec{F}$ is path-independent, so $\vec{F}$ is a gradient field and thus has a potential function.

Why Are Holes in the Domain of the Vector Field Important?

Example:

Let $\vec{F}$ be the vector field given by $\vec{F} = \dfrac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$.

(a) Calculate $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$. Does the curl test imply that $\vec{F}$ is path-independent?

(b) Calculate $\oint_C \vec{F} \cdot d\vec{r}$, where C is the unit circle centered at the origin and oriented counterclockwise. Is $\vec{F}$ a path-independent vector field?

(c) Explain why your answers to parts (a) and (b) do not contradict Green's Theorem.

The Curl Test for Vector Fields in $\mathbb{R}^3$:

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ is a vector field in $\mathbb{R}^3$, we define a new vector field, curl $\vec{F}$, or $\nabla \times \vec{F}$ in $\mathbb{R}^3$ by

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}.$$

THE CURL TEST FOR VECTOR FIELDS IN $\mathbb{R}^3$:
Suppose $\vec{F}$ is a vector field in $\mathbb{R}^3$ with continuous partial derivatives and such that

- The domain of $\vec{F}$ has the property that every closed curve in it can be contracted to a point in a smooth way, staying at all times within the domain.
- $\nabla \times \vec{F} = \vec{0}$.

Then $\vec{F}$ is path-independent, so $\vec{F}$ is a gradient field and has a potential function.

Examples:

2. Decide if the given vector field is the gradient of a function f.

   (a) $\vec{F} = 2xy\vec{i} + x^2\vec{j}$

   (b) $\vec{F} = (2xy^3 + y)\vec{i} + (3x^2y^2 + x)\vec{j}$

(c) $\vec{F} = 2x\cos(x^2+z^2)\vec{i} + \sin(x^2+z^2)\vec{j} + 2z\cos(x^2+z^2)\vec{k}$.

3. Find $\oint_C \vec{F}\cdot d\vec{r}$ if $\vec{F}(x,y) = -y^3\vec{i} + x^3\vec{j}$ and C is the circle of radius 3, centered at the origin, oriented counterclockwise.

4. Calculate $\oint_C ((3x+5y)\vec{i} + (2x+7y)\vec{j})\cdot d\vec{r}$ where C is the circular path with center (a,b) and radius m, oriented counterclockwise.

5. Show that the line integral of $\vec{F} = x\vec{j}$ around a closed curve in the xy-plane, oriented as in Green's Theorem, measures the area of the region enclosed by the curve.