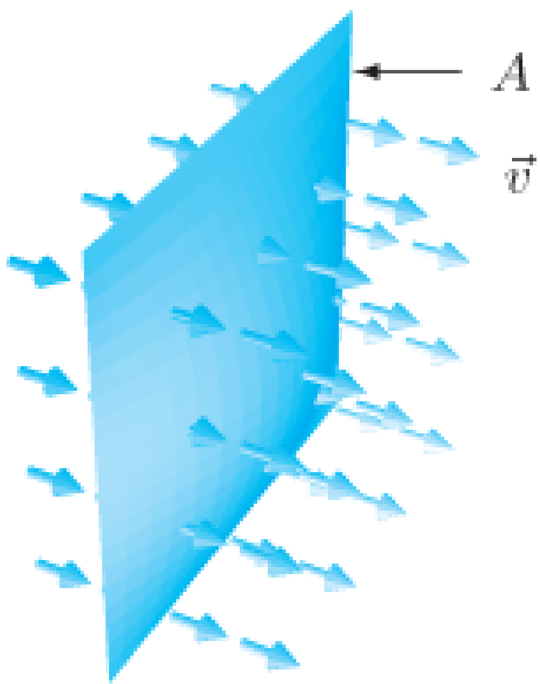
Section 19.1: The Idea of a Flux Integral

Flow Through a Surface

One of the most intuitive ways to begin to think about flux is with in terms of water flowing through a stretched out net. Imagine water flowing through a stretched out fishing net across a stream, and suppose you were interested in measuring the flow rate of water through the net. In other words, what is the volume of water passing through the net per unit time?

Example: A flat square surface of area A, in m^2, is immersed in a fluid. The fluid flows with a constant velocity $\vec{v}$, in m/s, perpendicular to the square. Write an expression for the rate of flow, in m^3/s.



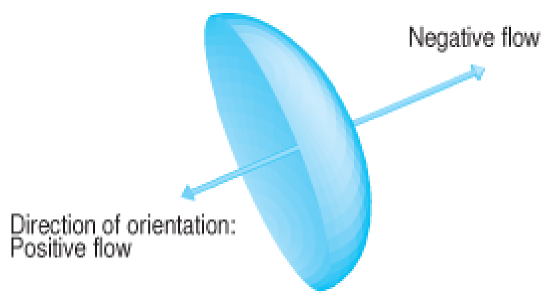
The flow rate from above is called the *flux* of the fluid through the surface. We can also compute the flux of other vector fields, such as the electric field and the magnetic field. In these cases, there is no literal flow taking place, yet the above example still provides valuable intuition.

Orientation of a Surface:

It is necessary for us to decide which direction of flow is going to be given a positive flux, and which direction of flow is going to be assigned a negative flux. Therefore we introduce the notion of *orientation of a surface:*

At each point on a smooth surface there are two unit normals, one in each direction. *Choosing an orientation* means picking one of these normals at every point on the surface in a continuous way. The unit normal vector in the direction of the orientation is denoted by $\vec{n}$. For a closed surface (that is, the boundary of a solid region), we choose the *outward orientation* unless otherwise specified.

We will say that the flux through a piece of a surface is positive if the flow is in the same direction as the orientation and negative if it is in the opposite direction.



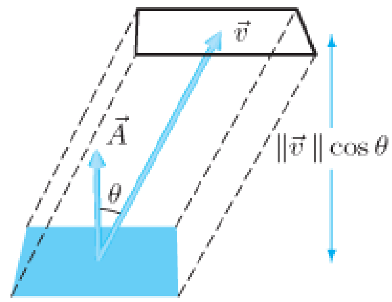
The Area Vector:

Since the flux through a surface depends on both the area of the surface and the orientation of the surface, it is useful to define a vector that encompasses both of these pieces of information.

The *area vector* of a flat, oriented surface is a vector $\vec{A}$ such that

- The magnitude of $\vec{A}$ is the area of the surface.
- The direction of $\vec{A}$ is the direction of the orientation vector $\vec{n}$.

The Flux of a Constant Vector Field Through a Flat Surface

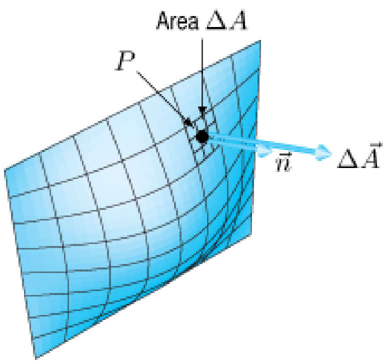


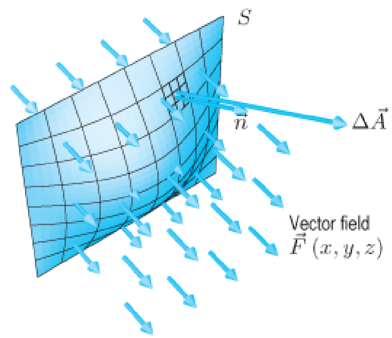
If $\vec{v}$ is constant and $\vec{A}$ is the area vector of a flat surface, then

$$\text{Flux through surface} = \vec{v} \cdot \vec{A}.$$

The Flux Integral

In the event that the vector field $\vec{F}$ is not constant and the surface S is not flat, we can play the same old game in order to be able to define the flux through the surface. That is, we cut up the surface S into a patchwork of smaller surfaces, each of which is almost flat.





On each small surface with area ΔA, we choose a unit orientation vector $\vec{n}$ and define the area vector to be $\Delta\vec{A} = \vec{n}\Delta A$.

If the patches are small enough, then it follows that $\vec{F}$ is approximately constant on each piece. Finish the development of the flux through the surface from there.

The *flux integral* of the vector field $\vec{F}$ through the oriented surface S is

$$\int_S \vec{F} \cdot d\vec{A} = \lim_{\|\Delta \vec{A}\| \to 0} \sum \vec{F} \cdot \Delta \vec{A}.$$

If S is a closed surface oriented outward, we describe the flux through S as the flux out of S.

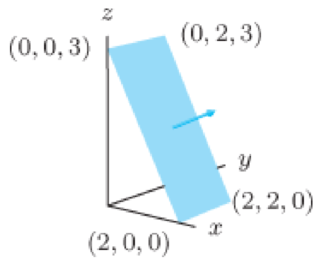
Examples:

1. Find the area vector of the circular disk of radius 5 in the xy-plane, oriented upward.

2. Compute $\int_S (2\vec{i} + 3\vec{k}) \cdot d\vec{A}$, where S is the disk of radius 4 perpendicular to the x-axis, centered at $(5, 0, 0)$ and oriented toward the origin.

3. Find the flux of the vector field $\vec{v} = \vec{i} - \vec{j} + 3\vec{k}$ through a disk of radius 2 in the xy-plane, oriented upward.

4. Compute the flux of $\vec{v} = \vec{i} + 2\vec{j} - 3\vec{k}$ through the rectangular region depicted below.



5. An electric charge q is placed at the origin in $\mathbb{R}^3$. The resulting electric field, $\vec{E}(\vec{r})$ at the point with position vector $\vec{r}$ is given by

$$\vec{E}(\vec{r}) = q\frac{\vec{r}}{\|\vec{r}\|^3}, \qquad \vec{r} \neq \vec{0}.$$

Find the flux out of the sphere of radius R centered at the origin.