Section 19.2: Flux Integrals For Graphs, Cylinders, and Spheres

Flux of a Vector Field Through the Graph of $z = f(x, y)$

If the surface S is the graph of a function $z = f(x, y)$, oriented upward, and if $\vec{F}$ is a smooth vector field, then we can determine the formula as before, by cutting up the surface S into small pieces of area vector $\Delta\vec{A}$ and defining the flux according to

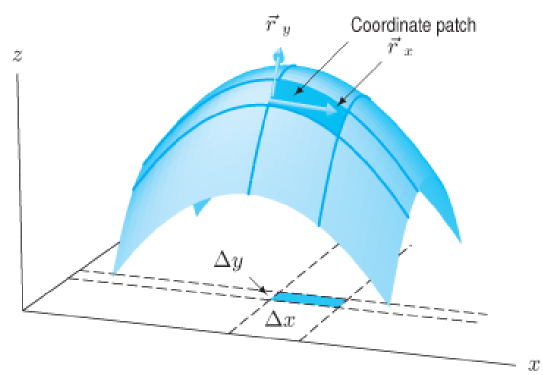
$$\int_S \vec{F} \cdot d\vec{A} = \lim_{\|\Delta\vec{A}\| \to 0} \sum \vec{F} \cdot \Delta\vec{A}.$$

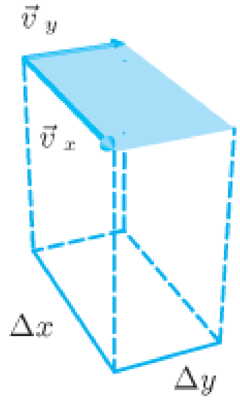
Now the question boils down to how do we divide S into small pieces? The easiest way is to use coordinate patches on the xy-plane with area $\Delta x \Delta y$, and to look at the corresponding patches on S.

The Area Vector of a Coordinate Patch

Recall that the cross product $\vec{v} \times \vec{w}$ has magnitude equal to the area of the parallelogram formed by $\vec{v}$ and $\vec{w}$, and points in a direction perpendicular to that parallelogram, given by the right hand rule. Therefore, we have

Area Vector of a Parallelogram $= \vec{A} = \vec{v} \times \vec{w}$





In order to find $\vec{v}_x$ and $\vec{v}_y$ (so that we can find the area vector of a coordinate patch), we note that at each point on the surface S there is a position vector $\vec{r} = x\vec{i} + y\vec{j} + f(x, y)\vec{k}$. Therefore, a cross section of S with fixed y has a tangent vector

$$\vec{r}_x = \vec{i} + f_x(x, y)\vec{k},$$

and a cross section of S with fixed x has a tangent vector

$$\vec{r}_y = \vec{j} + f_y(x, y)\vec{k}.$$

Now using the fact that $\vec{v}_x = \Delta x\,\vec{r}_x$ and $\vec{v}_y = \Delta y\,\vec{r}_y$, come up with a formula for the flux through $z = f(x, y)$.

THE FLUX OF $\vec{F}$ THROUGH A SURFACE GIVEN BY THE GRAPH OF $z = f(x, y)$:

Suppose that the surface S is the part of the graph of $z = f(x, y)$ above the region R in the xy-plane, oriented upward. The flux of $\vec{F}$ through S is

$$\int_S \vec{F} \cdot d\vec{A} = \int_R \vec{F}(x, y, f(x, y)) \cdot \left(-f_x\vec{i} - f_y\vec{j} + \vec{k}\right)\, dx\, dy$$

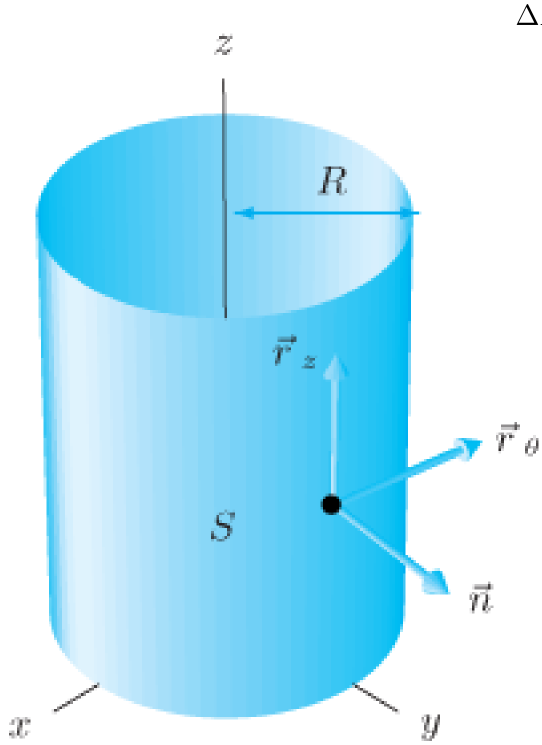
Examples:

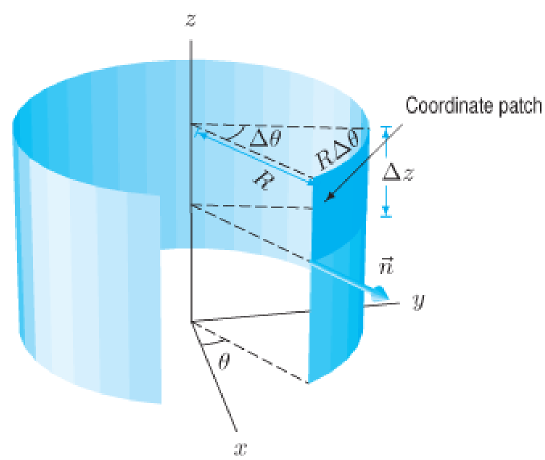
1. Calculate the flux of $\vec{F} = x\vec{i} + y\vec{j}$ through the surface S given by $z = 25 - (x^2 + y^2)$ and above the disk of radius 5 centered at the origin, oriented upward.

Flux of a Vector Field Through a Cylindrical Surface

If S is a cylinder of radius R, centered along the z-axis, and oriented away from the z-axis, then a coordinate patch has surface area

$$\Delta A \approx R\,\Delta\theta\,\Delta z.$$





Question: How do we characterize the outward unit normal?

We arrive at the following characterization for the infinitesimal area vector, $d\vec{A}$:

$$d\vec{A} = (\cos\theta\vec{i} + \sin\theta\vec{j})R\,dz\,d\theta.$$

THE FLUX OF A VECTOR FIELD THROUGH A CYLINDER

The flux of $\vec{F}$ through the cylindrical surface S, of radius R, and oriented away from the z-axis, is given by

$$\int_S \vec{F}\cdot d\vec{A} = \int_T \vec{F}(R,\theta,z)\cdot(\cos\theta\vec{i} + \sin\theta\vec{j})R\,d\theta\,dz,$$

where T is the θz-region corresponding to S.

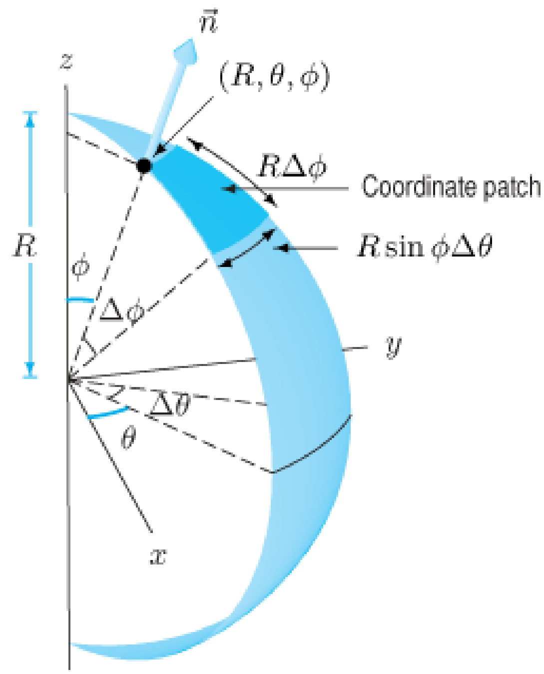
Examples:

2. Calculate The flux of $\vec{F} = x\vec{i} + 2y\vec{j} + 3z\vec{k}$ through the cylindrical surface S, of radius 10, from $0 \le z \le 5$, centered along the z-axis, and oriented away from the z-axis.

Flux of a Vector Field Through a Spherical Surface

As is the case for cylinders, it is easy to use spherical coordinates to get an idea of what a small piece of area, ΔA, should look like on a sphere of radius R. In this case we have

$$\Delta A \approx R^2 \sin\phi \, \Delta\phi \, \Delta\theta$$



Problem: Using the same ideas as we used for the cylindrical surface, find a form for an outward unit normal $\vec{n}$ to the surface of the sphere, and therefore a general form for the area vector $d\vec{A}$.

THE FLUX OF A VECTOR FIELD THROUGH A SPHERE:

The flux of $\vec{F}$ through the spherical surface S, with radius R and oriented away from the origin, is given by

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{A} &= \int_S \vec{F} \cdot \frac{\vec{r}}{\|\vec{r}\|}\, dA \\ &= \int_T \vec{F}(R, \theta, \phi) \cdot (\sin\phi\cos\theta\vec{i} + \sin\phi\sin\theta\vec{j} + \cos\phi\vec{k}) R^2 \sin\phi\, d\phi\, d\theta,\end{aligned}$$

where T is the $\phi\theta$-region corresponding to S.

Examples:

3. Find the flux of $\vec{F} = z\vec{k}$ through S, the upper hemisphere of radius 2 centered at the origin, oriented outward.

4. Find the flux of $\vec{F} = \cos(x^2 + y^2)\vec{k}$ through S, the part of the surface $z = 25 - (x^2 + y^2)$ above the disk of radius 5 centered at the origin, oriented upward.

5. Find the flux of $\vec{F} = -xz\vec{i} - yz\vec{j} + z^2\vec{k}$ through S, the cone $z = \sqrt{x^2 + y^2}$ for $0 \leq z \leq 6$, oriented upward.