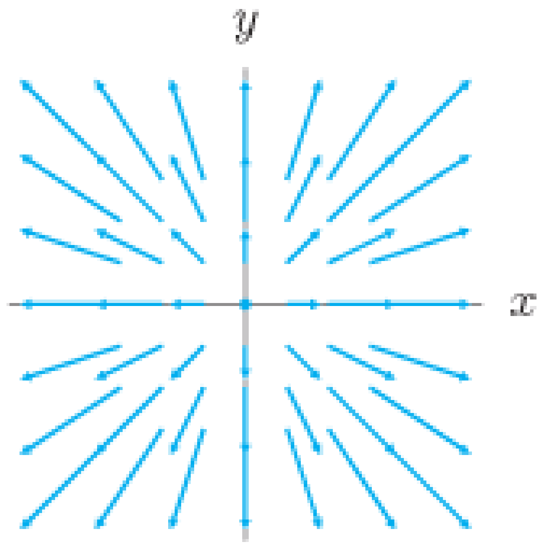
Section 19.3: The Divergence of a Vector Field

Question:

Consider the figure below.



Imagine that you were interested in trying to find a way to quantify, for lack of a better phrase, the "amount of flux emanating from a single point" on a vector field like the one above. It is clear that the vector field is radiating away from the origin, so we should imagine that the origin is a point from which a lot of flux emanates. How can you make this idea concrete?

Definition of Divergence

GEOMETRIC DEFINITION OF DIVERGENCE:
The *divergence*, or *flux density*, of a smooth vector field $\vec{F}$, written $\operatorname{div}\vec{F}$, is a scalar-valued function defined by

$$\operatorname{div}\vec{F}(x,y,z) = \lim_{\text{Volume}(S)\to 0} \frac{\int_S \vec{F}\cdot d\vec{A}}{\text{Volume(S)}}.$$

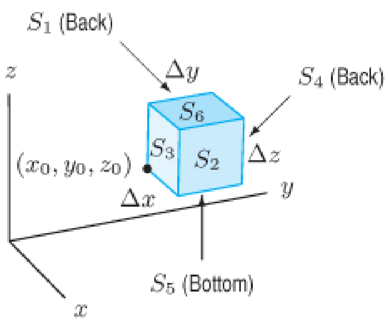
Here S is a sphere centered at (x,y,z), oriented outward, that contracts down to (x,y,z) in the limit.

CARTESIAN COORDINATE DEFINITION OF DIVERGENCE:
If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Example: Using the geometric definition of divergence, compute the divergence $\vec{F}(\vec{r}) = \vec{r}$ at the origin.

Why Do the Two Definitions Give the Same Result



Examples:

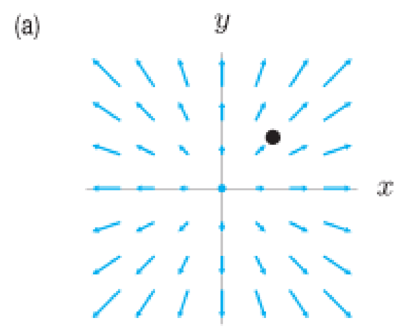
1. The vector field $\vec{E}$ is defined as

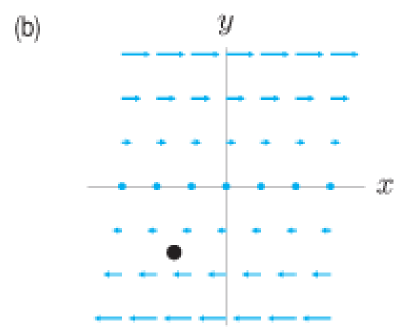
$$\vec{E} = \frac{\vec{r}}{\|\vec{r}\|^p},$$

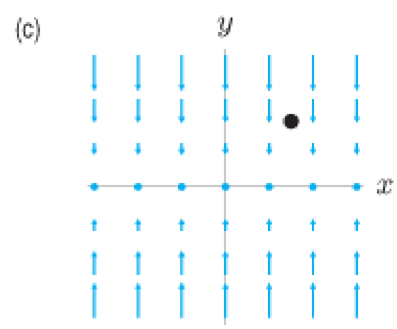
where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{r} \neq \vec{0}$.

For what value(s) of p is $\vec{E}$ divergence free everywhere it is defined?

2. For each of the following vector fields, sketched in the xy-plane, decide if the divergence is positive, negative, or zero at the indicated point.







3. A smooth vector field $\vec{F}$ has $\operatorname{div}\vec{F}(1,2,3) = 5$. Estimate the flux of $\vec{F}$ out of a small sphere of radius 0.01 centered at the point $(1,2,3)$.

4. The flux of $\vec{F}$ out of a small sphere of radius 0.1 centered at $(4,5,2)$, is 0.0125. Estimate $\operatorname{div}\vec{F}$ at $(4,5,2)$.