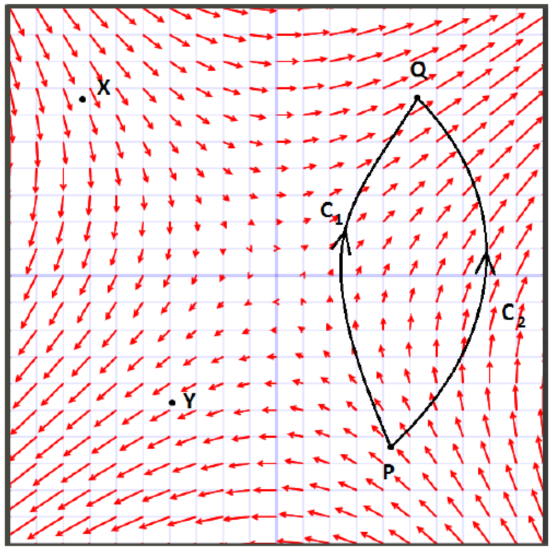
Team Homework 10

1. The picture below shows the gradient field $\overrightarrow{\nabla f}$ for a differentiable function $f(x, y)$. Shown in the picture are four points P, Q, X, and Y, and two paths C_1 and C_2 from P to Q.[1]



   (a) Is $\int_{C_1} \overrightarrow{\nabla f} \cdot d\vec{r}$ greater than, less than, or equal to $\int_{C_2} \overrightarrow{\nabla f} \cdot d\vec{r}$? Justify your answer.

   (b) Arrange the following quantities from least to greatest: $f(X)$, $f(Y)$, and $f(0, 0)$. If two quantities are equal, note this in your ordering. Justify your answer.

2. Evaluate the following line integrals. For each problem, try to come up with the most efficient strategy for computing the integral.

   (a) The line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = x^2\,\vec{i} + 4y\,\vec{j}$, and C is the part of the curve $y = 4/2^x$ from $(0, 4)$ to $(2, 1)$.

   (b) The line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = 3y\,\vec{i} - 4x\,\vec{j}$, and C is a semicircle of radius 2 centered at the origin, from $(0, -2)$ to $(2, 0)$ to $(0, 2)$.

   (c) The line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = \frac{2x}{x^2 + y^2 + z^2}\,\vec{i} + \frac{2y}{x^2 + y^2 + z^2}\,\vec{j} + \frac{2z}{x^2 + y^2 + z^2}\,\vec{k}$, and C is the upward spiral around the z-axis that begins at $(1, 0, 0)$ and ends at $(1, 0, 4)$, wrapping around the z-axis twice.

[1] The picture was created using an online vector field grapher at http://kevinmehall.net/p/equationexplorer/vectorfield.html.

3. Let C_1 be the parametrized path given by $\vec{r}_1(t) = t\cos(2\pi t)\,\vec{i} + t\sin(2\pi t)\,\vec{k}$ for t in $[0, 2]$, and let C_2 be the parametrized path given by $\vec{r}_2(t) = t\cos(2\pi t)\,\vec{i} + t\,\vec{j} + t\sin(2\pi t)\,\vec{k}$ for t in $[0, 2]$.

(a) Describe the motion of a particle moving along each of these paths, and sketch each path.

(b) Evaluate $\displaystyle\int_{C_2} \vec{F} \cdot d\vec{r}$ for the vector field $\vec{F}(x, y, z) = yz\,\vec{i} + z(x+1)\,\vec{j} + (xy + y + 1)\,\vec{k}$.

(c) Give an example of a nonzero vector field $\vec{G}$ such that

$$\int_{C_1} \vec{G} \cdot d\vec{r} = \int_{C_2} \vec{G} \cdot d\vec{r}.$$

Justify your answer.

(d) Give an example of two different, nonzero vector fields $\vec{H}_1$ and $\vec{H}_2$ such that

$$\int_{C_1} \vec{H}_1 \cdot d\vec{r} = \int_{C_1} \vec{H}_2 \cdot d\vec{r}.$$

Justify your answer.