Exam 2 Study Guide

Math 223 Instructor: Dr. Gilbert

1. Consider the function $f(x, y) = x^2 e^{xy}$ and the point $P = (3, 0, 9)$.

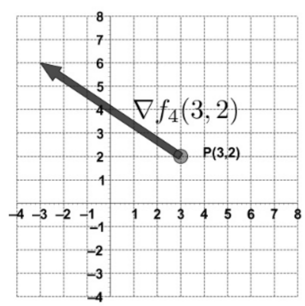
   (a) Find an equation of the tangent plane to the graph of $z = f(x, y)$ at the point P.

   (b) Use differentials to approximate $f(2.9, 0.2)$.

2. Find or estimate, depending on the type of data provided, the partial derivatives f_x and f_y at the point $(3, 2)$ for the following functions:

   (a) The function f_1 is given by the formula
   $$f_1(x, y) = x^2 \cos(xy)$$

   (b) The only thing we know about f_4 is its gradient vector at $(3, 2)$



3. A gnat with a keen grasp of multivariable calculus notes that the relative humidity in the greenhouse in which it is flying is given by $H(x, y, z) = \frac{1}{2}\sin(xy) + \frac{1}{z+1}$.

   (a) In what direction should the gnat fly from its current position of $(\pi, 2, 1)$ to decrease the humidity of the air around it the fastest?

   (b) How does the humidity change if the gnat intead flies in the direction of the point $(\pi + 3, 2, 5)$? Give your answer as a rate of change in this direction.

4. Suppose that three quantities x, y, and z are related to each other by the equation $2x^2+3y^2+z^2 = 20$. The graph of this equation is a surface S in $\mathbb{R}^3$.

   (a) Verify that the point $P = (2, 1, 3)$ is a point on S and find an equation of the tangent plane to S at P.

   (b) Near P, we can think of S as the graph of a function $z = f(x, y)$. Without finding $f(x, y)$ explicitly, determine its linear approximation $L(x, y)$ near $x = 2$, $y = 1$.

   (c) Approximate the value of z corresponding to $x = 1.97$ and $y = 1.12$.

5. The temperature T of a mole of oxygen is given (in suitable units) in terms of its pressure P and volume V by the equation

$$T = \frac{16}{V} - \frac{1}{V^2} - P + 11VP.$$

   (a) Find the temperature T and the differential dT if $V = 1$ and $P = 10$.

   (b) Suppose that the pressure increases to $P = 10.1$. Use your answer from part (a) to estimate how much the volume would have to change for the temperature to remain constant. Does the volume have to increase or decrease?

6. Throughout this problem,

$$f(x, y, z) = x^2 + 2y^2 + \frac{z^2}{2} + 2xy.$$

   The level surfaces of this function are ellipsoids.

   (a) Find the directional derivative of f at $(-1, 1, 2)$ in the direction of the vector $3\hat{\imath} + 4\hat{k}$.

   (b) Find the direction (unit vector) in which the rate of change of f is greatest at the point $(-1, 1, 2)$, and the rate of change (scalar) in this direction.

   (c) Find all of the points on the level surface $f(x, y, z) = 4$ where the tangent plane to the surface is parallel to the xz-plane.

7. Throughout this problem, let

$$f(x, y, z) = \cos(xy) + z^2.$$

   (a) Compute $\vec{\nabla} f$ at an arbitrary point (x, y, z).

   (b) Find the directional derivative of f at $(1, \pi/2, 1)$ in the direction of the vector $4\hat{\imath} - 3\hat{k}$.

   (c) Find the equation of the tangent plane to the surface $\cos(xy)+z^2 = 1$ at the point $(1, \pi/2, 1)$.

8. Consider the function $f(x, y) = y^2 - x^2 - yx^3$.

   (a) Compute $\vec{\nabla} f(1, -1)$.

   (b) Find an equation of the tangent plane to $z = f(x, y)$ at the point $(1, -1, 1)$.

   (c) Compute the directional derivative of f at the point $(1, -1)$ in the direction of $3\hat{\imath} + \hat{\jmath}$.

   (d) True or false: The level curve defined by $f(x, y) = 1$ has, at the point $(1, -1)$, tangent line parallel to $3\hat{\imath} + \hat{\jmath}$. Justify your answer.

9. Let $f(x, y) = x^2 + y^2 - xy$. Find the critical points of f.

10. Let $h(x, y) = e^{x-y}$.

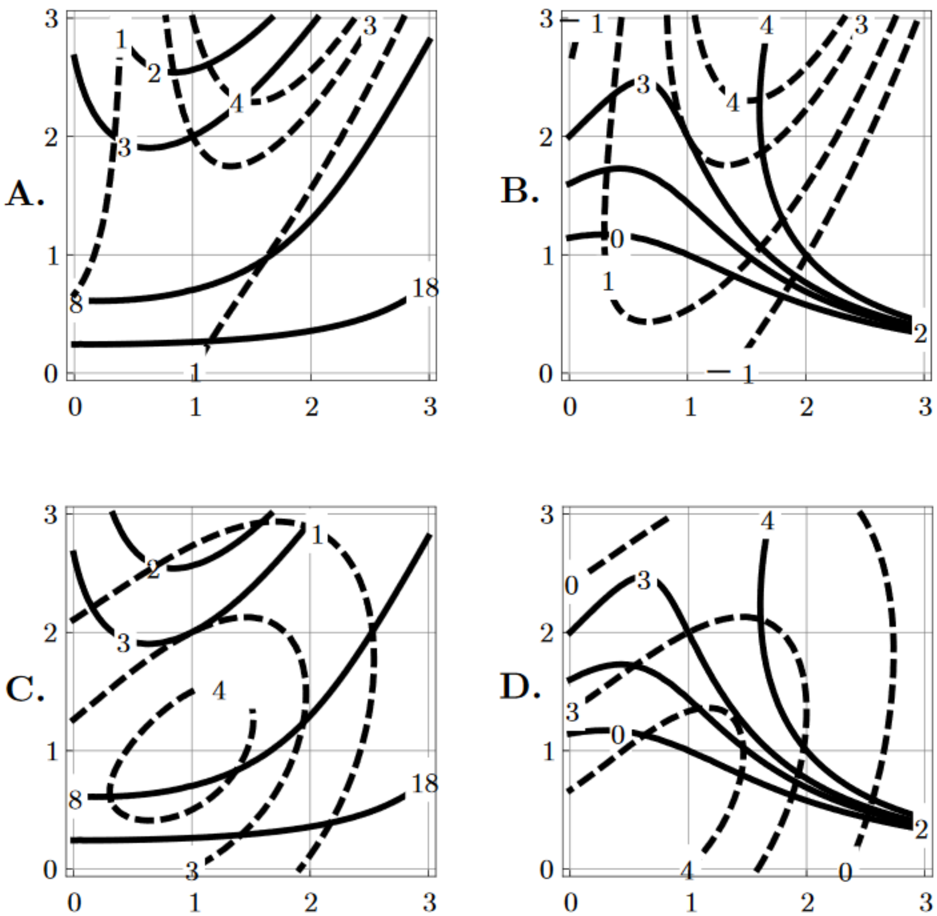
   (a) Suppose that $x = f(t)$ and $y = g(t)$, where f and g are differentiable functions of t satisfying $f(3) = g(3) = 1/2$, and $f'(3) = 2$ while $g'(3) = 4$. Compute $\dfrac{dh}{dt}$ when $t = 3$.

   (b) Suppose now that $x = f(s, t)$ with $f(3, 2) = 7$, $g(3, 2) = -5$, $f_s(3, 2) = 8$, $f_t(3, 2) = 1$, $g_s(3, 2) = 5$ and $g_t(3, 2) = -1$. Compute the partial of h with respect to t at the point $(3, 2)$.

11. Sketch a solid whose volume could be given by the integral

$$\int_0^2 \int_0^1 (5 - 2x - y)\, dy\, dx$$

12. If the linear approximations of two functions $f(x, y)$ and $g(x, y)$ at the point $(1, 2)$ both turn out to be
$$L(x, y) = 3 + (x - 1) - 2(y - 2),$$
which of the figures below could give the level curves of f (solid) and g (dashed)? Briefly explain your answer.



13. The gravitational force on an object at a distance r from the center of the earth is given by

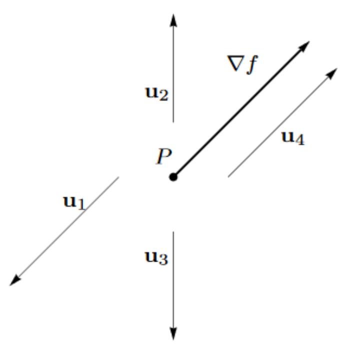
$$F = \frac{GMm}{r^2},$$

where M and m are the masses of the Earth and object, respectively, and G is a constant. Note that as a rocket moves into space it loses mass because it is burning fuel, so that m and r both change. Use differentials to write an expression for the change in the gravitational force on a rocket as it moves from $r = 700,000$ m to $r = 704,000$ m and its mass decreases from $m = 600,000$ kg to $m = 590,0000$ kg. Express your answer in terms of G and M.

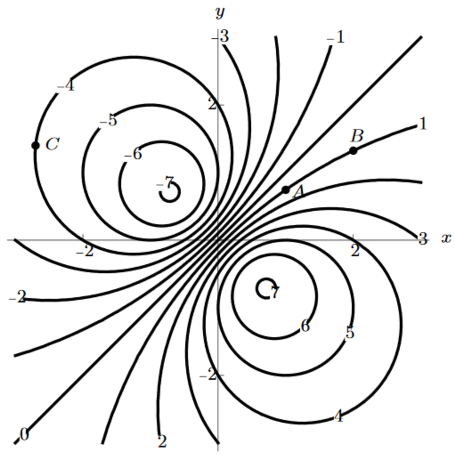
14. The figure below shows the gradient of a function f at a point P, along with four unit vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$. Arrange, in increasing order, the following quantities:

$$\begin{array}{lll} f_{\mathbf{u}_1}(P) & f_{\mathbf{u}_1}(P) & f_{\mathbf{u}_3}(P) \\ f_{\mathbf{u}_4}(P) & \|\vec{\nabla} f(P)\| & \text{The number } 0 \end{array}$$

Briefly explain how you determine your ordering.



15. A contour diagram for a function $f(x, y)$ is shown in the figure below.



(a) On the graph, sketch a reasonably accurate estimate of the gradient $\vec{\nabla} f$ at each of the points A, B, and C.

(b) If $f(0, -2) = 4$ and $\vec{\nabla} f(0, -2) = \frac{4}{5}\hat{\imath} + \frac{12}{25}\hat{\jmath}$, write an equation for the tangent plane to the surface $z = f(x, y)$ at $(0, -2)$.

16. Let $R(x, y)$ be a function of two variables such that $R(3, -3) = 4$ and $\vec{\nabla} R(3, -3) = -\hat{\imath} + 2\hat{\jmath}$. Write an equation for the tangent plane to the graph of R at the point $(3, -3, 4)$.

17. Let a point on Earth close to London, England be expressed by the coordinates (x, y), where x and y are the longitude and latitude, respectively, of the point. Suppose that the temperature of a place close to London is given by the function $T(x, y) = 5 \sin x - \frac{3y}{2} + 110$.

(a) The coordinates of the city Greenwich, England are $(0, 101/2)$. In which direction should one walk from this point such that the temperature will increase the fastest? Give your answer in the form of a unit vector that points in the direction.

(b) What is the directional derivative of T at Greenwich, in the direction towards Aberdeen, Scotland, which has coordinates $(2, 115/2)$?

(c) Suppose that the coordinates of an African swallow migrating south t hours after it has taken flight are given by $(x(t), y(t))$, where

$$x(t) = (t - 3)^3, \qquad y(t) = 52 - \frac{t}{2}.$$

How fast is the temperature changing for the bird when it is directly over Greenwich?

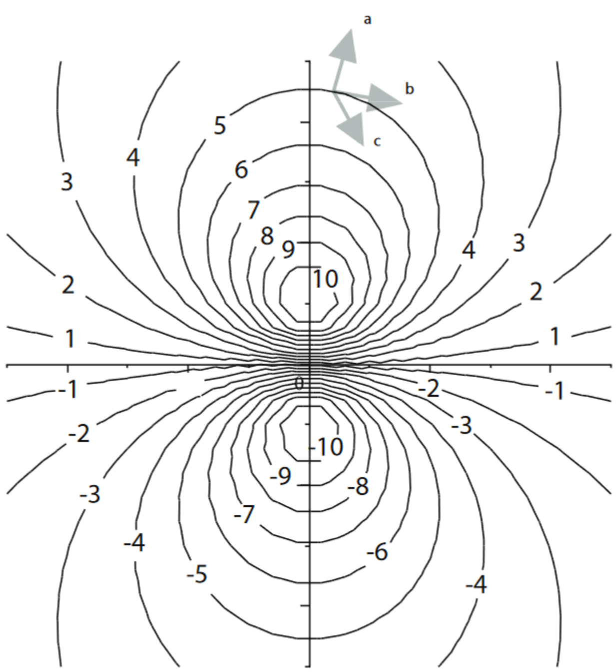
18. Let $f(x, y) = 6x + 2y^3 + 3y^2 - 3x^2$. Decide whether the following points correspond to a local maximum for f, a local minimum for f, a saddle point for f, or neither.

    (a) $(1, -1)$

    (b) $(0, -1)$

    (c) $(1, 0)$.

19. The figure below shows some level curves for the function $g(x, y)$ in dark ink. In light ink are three planar unit vectors **a**, **b** and **c**, which have been drawn at the point $(1, 9)$.



    (a) In the above picture, please mark clearly the location of all critical points for the function $g(x, y)$. If the critical point is a local maximum, mark it with the letter T. If the critical point is a local minimum, mark it with the letter B. If the critical point is a saddle point, mark it with the letter S.

    (b) Of the directional derivatives, $g_{\mathbf{a}}(1, 9)$, $g_{\mathbf{b}}(1, 9)$, and $g_{\mathbf{c}}(1, 9)$, determine which is the least, which is the greatest, and which has a value in between the other two.

20. The radius of a cylinder is increasing at a constant rate of 2 inches per second and the height of the cylinder is decreasing at a constant rate of 2 inches per second.

    (a) Is the volume of the cylinder increasing or decreasing at the time when the radius is 10 inches and the height is 10 inches?

    (b) Suppose that 3 seconds had passed from the time of question (a). Is the volume of the cylinder increasing or decreasing at that time?

21. Suppose that the temperature at a point (x, y, z) in space is given by

$$T(x, y, z) = 100e^{-x^2-2y^2-3z^3}.$$

    (a) Find the rate of change of temperature at the point $(2, 1, 0)$ in the direction *toward* the point $(3, 3, 3)$.

    (b) In which direction does the temperature decrease the most rapidly at the point $(2, 1, 0)$? Describe the direction using a unit vector.

22. Find an equation for the tangent plane to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(1, 2, 3)$.

23. Find an equation for the tangent plane to the surface given by the equation $x^2 + y^2 - z^2 = 12$ at the point $(2, 3, 1)$.

24. Let $u(x, t) = \sin(x - t)$. Which of the equations does this function satisfy?

    (a) $u_t = u$

    (b) $u_t = u_x$

    (c) $u_t u_x = 0$

    (d) $u_{tx} = 0$

    (e) $u_t = u_{xx}$

    (f) $u_{tt} + u_{xx} = 0$

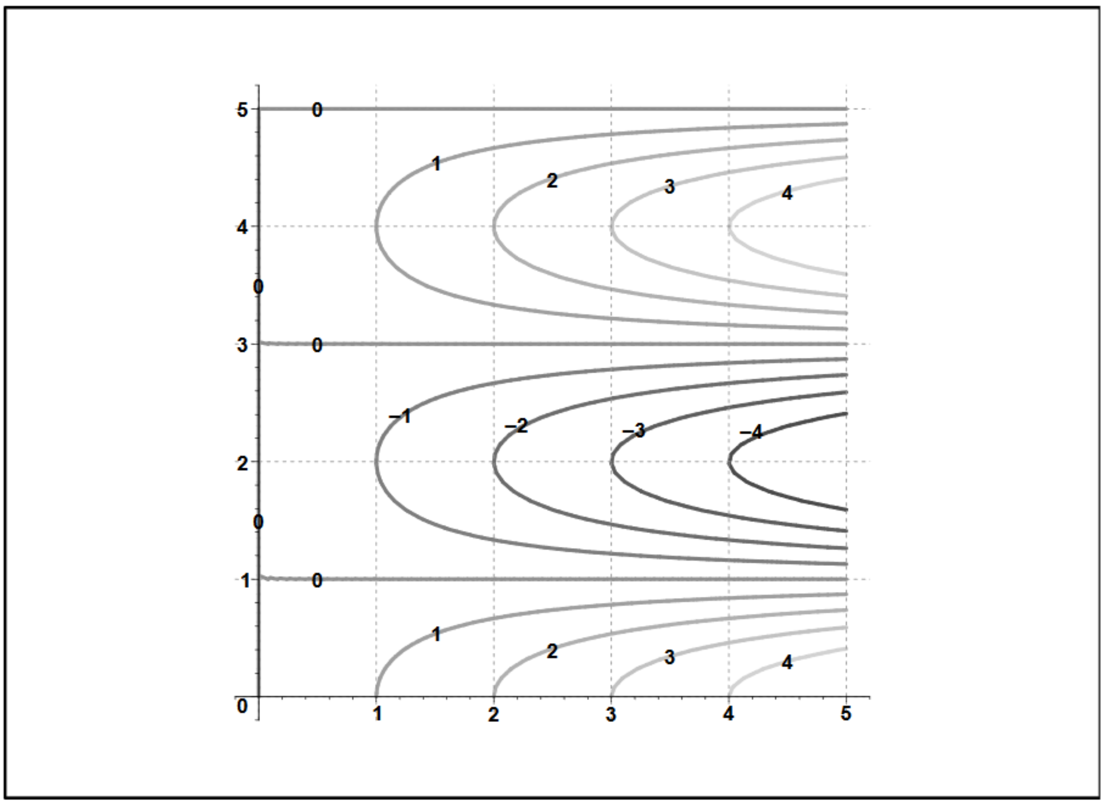
    (g) $u_{tt} - u_{xx} = 0.$

24. Consider the function $f(x,y,z) = x^2 + \dfrac{y^2}{2} + 2z^2 + 2xz$. Find all points on the level surface $f(x,y,z) = 4$ at which the tangent plane is parallel to the xy-plane.

25. Find and classify all critical points of the function

$$f(x,y) = 5x^2y - 2xy^2 + 30xy - 3.$$

26. Using the level curves of this function on the plot below, estimate $\displaystyle\int_R f\,dA$, where R is the rectangular region $0 \le x \le 5$, $0 \le y \le 5$.



27. Find the exact value of the integral $\int_R f\,dA$, where R is the rectangular region $0 \le x \le 5$, $0 \le y \le 5$, and $f(x, y) = x\cos\left(\frac{\pi y}{2}\right)$.

28. Sketch the region of integration and evaluate the integral

$$\int_0^2 \int_{2y}^4 \sin(x^2)\,dx\,dy$$

by changing the order of integration.

29. Sketch the region of integration and evaluate the integral

$$\int_0^2 \int_{\sqrt{y/2}}^1 ye^{x^5}\,dx\,dy$$

30. Find and classify all critical points of the function

$$g(x, y) = \sin x \cos y$$

on the square $-1 \le x \le 4$, $-1 \le y \le 4$.

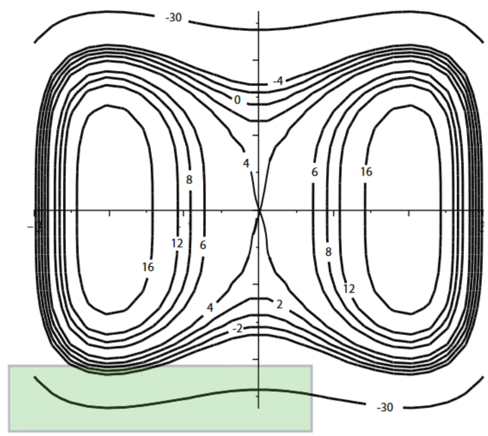
31. Consider the integral

$$\int_{-2}^0 \int_0^{\sqrt{4-y^2}} f(x, y)\,dx\,dy + \int_0^2 \int_{y-2}^{\sqrt{4-y^2}} f(x, y)\,dx\,dy.$$

Sketch the region of integration and rewrite the integral by reversing the order of integration.

32. Consider the function $f(x, y)$, whose level curves are shown below. Call the shaded rectangular region D. Assume that each hash mark on the axes represents an integer. Arrange the following integrals from least to greatest:

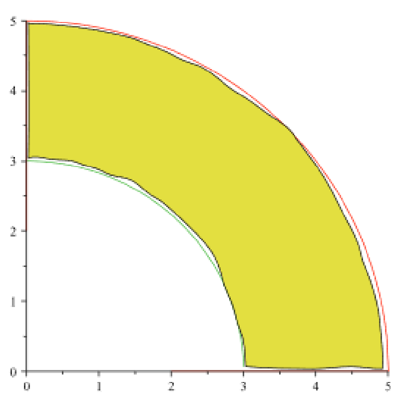
$$\int_{-1}^{1}\int_{-5}^{-3} f(x,y)\,dx\,dy, \qquad \int_{3}^{5}\int_{0}^{1/2} f(x,y)\,dy\,dx, \qquad \int_D f\,dA.$$



33. Consider the lamina which lies entirely in the first quadrant of the xy-plane and is bounded by the curves $y = \sqrt{9 - x^2}$, $x = \sqrt{25 - y^2}$, $x = 0$ and $y = 0$. Suppose that the density of the lamina at a given point is given by

$$\rho(x, y) = \frac{xy}{x^2 + y^2}.$$

Set up, but *do not evaluate*, an integral representing the mass of the lamina in Cartesian coordinates. You may not be able to write the mass as a single integral, but will be able to write it instead as the sum of two integrals.



34. Evaluate the integral $\int_D y\,dA$, where D is the region in the first quadrant bounded by the parabolas $x = 8 - y^2$ and $x = y^2$.

35. Evaluate the integral
$$\int_D \frac{x}{x^2+y^2}\,dA,$$
where D is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

36. Evaluate the integral
$$\int_0^1 \int_x^1 e^{x/y}\,dy\,dx.$$

37. Suppose that the density function of the square lamina, $0 \le x \le 1$, $0 \le y \le 1$, is given by
$$\rho(x,y) = \int_x^1 \cos(t^2)\,dt.$$
Evaluate the mass of the lamina.