Exam 3 Study Guide

Math 223 Instructor: Dr. Gilbert

1. (a) Find parametric equations of the line of intersection of the planes $x - 3y + 2z = -1$ and $4x + y + 7z = 9$.

   (b) Find parametric equations of the line through the points $(3, -1, 2)$ and $(5, 1, 3)$.

2. (a) Find parametric equations of the tangent line to the curve $\vec{r}(t) = \langle 2 + t^3, 1 - 4t, 5 - t^2 \rangle$ at the time $t = 1$.

   (b) Find the point of intersection of this line with the xy-plane.

3. A particle moves so that $\vec{r}(t) = (3\cos t + 3t \sin t)\hat{\imath} + (3\sin t - 3t\cos t)\hat{\jmath} + (2t^2)\hat{k}$. Find the distance travelled by the particle from $t = 0$ to $t = 4\pi$.

4. Consider the function $f(x, y) = x^2 e^{xy}$ and the point $P = (3, 0, 9)$.

   (a) Find an equation of the tangent plane to the graph of $z = f(x, y)$ at the point P.

   (b) Find parametric equations of the normal line to the graph of $z = f(x, y)$ going through the point P.

5. (a) Find an equation of the plane containing the line given by $\vec{r}(t) = (2+3t)\hat{\imath} + (1-2t)\hat{\jmath} + (-1+t)\hat{k}$ and the point $(1, 3, 1)$.

   (b) Find a parametric equation for the line of intersection of the planes $x - y + 2z = 2$ and $3x + y - z = 4$.

6. Let L and M be the lines with parametric equations $\vec{r}_1(t) = \langle -t + 1, 2t + 2, t + 3 \rangle$ and $\vec{r}_2(t) = \langle 2t, t+4, -t+4 \rangle$. Do these lines intersect? If they do, give the coordinates of the point of intersection.

7. A golf ball is hit at time $t = 0$ Its position vector as a function of time is given by
$$\vec{r}(t) = \langle 2t, 3t, -t^2 + 4t \rangle.$$
   Notice that at $t = 0$, the ball is at the origin of the coordinate system. The xy-plane represents the ground. At some time $t_1 > 0$ the ball will return to the xy-plane at some point $P = (a, b, 0)$.

   (a) Compute the velocity and the speed of the ball at an arbitrary time t.

   (b) Find the time t_1 and the coordinates of the point P where the ball hits the xy-plane again.

   (c) Find the equation of the vertical plane containing the trajectory.

8. Consider the parametric curve
$$\vec{r}(t) = \langle t, t^2, t^3 \rangle.$$
   (a) Find a parametric equation $\vec{q}(s)$ of the tangent line to the curve at time $t = T$.

   (b) If $T = 2$, find the point of intersection of the tangent line with the xy-plane.

9. If the vector function $\vec{r}_1(t) = \langle t+2, 2t^2 - t + 1, 3 + t - t^3 \rangle$ is at the point $(2, 1, 3)$ tangent to the plane containing the curve $\vec{r}_2(s) = \langle s^2 + 2, -s + 1, 2s^2 - s + 3 \rangle$, find an equation of the plane.

10. The spacecurve

$$\vec{r}(t) = (2t - \cos t + 3)\hat{\imath} + (\sin^2 t + 4t)\hat{\jmath} + \frac{1}{2}(-\cos^2 t + 2\cos t + 1)\hat{k}$$

lies in a plane. Find the equation of the plane.

11. Use projections to find the distance between the line given by parametric equations $x(t) = 2t$, $y(t) = t + 2$, and $z(t) = 3 - t$ and the plane $2x - y + 3z = 9$. Draw a figure that illustrates what your calculations are finding.

12. Consider the spacecurve given by parametric equations

$$x = -2t - 2, \qquad y = t^2 + 3t + 2, \qquad z = 2t + 2.$$

What is the unit tangent vector to this curve at the point $(-2, 2, 2)$?

13. Consider the parabolic cylinder

$$x = y^2$$

and the hyperbolic paraboloid

$$z = x^2 - y^2.$$

Write parametric equations for the curve formed by the intersection of the surfaces above.

14. A particle is traveling along the curve $\vec{r}(t) = 3\sin t\hat{\imath} + 3\cos t\hat{\jmath} + t\hat{k}$, where t is the time variable.
(a) Compute the velocity $\vec{v}(t)$ and speed $s(t)$ of the particle at an arbitrary time t.

(b) Find an equation for the tangent line L of the curve $\vec{r}(t)$ at the point $P = (0, 3, 0)$.

(c) Find the distance that the particle traveled from $t = 0$ to $t = 7\pi$.

(d) The position of another moving particle at time t is

$$\vec{r}_1(t) = \langle t - 2, t^2 - 1, 2 - t \rangle.$$

Will they meet? Explain.

15. The surface $\mathcal{S}$ contains two curves $\vec{r}_1(t)$ and $\vec{r}_2(t)$ given by parametric equations

$$\vec{r}_1(t) = \langle t - t^3 + 3, 1 - t + 2t^2, 2 + t \rangle$$

and

$$\vec{r}_2(t) = \langle t^3 - 2t + 4, t, 2 \rangle,$$

respectively. These curves intersect at the point $(3, 1, 2)$, which is, of course, on the surface $\mathcal{S}$. Find an equation of the tangent plane to the surface $\mathcal{S}$ at $(3, 1, 2)$.

16. A ball is thrown eastward into the air from the origin (in the direction of the positive x-axis) with the initial velocity $50\hat{\imath} + 80\hat{k}$. The speed is measured in feet per second. In addition to the gravity which gives rise to an acceleration of $-32\hat{k}$ ft/s^2, the southward wind also generates an acceleration of $-4\hat{\jmath}$ ft/s^2. Where does the ball land? Find the coordinates.

17. There is a function $f(x, y)$ which is differentiable whose exact formula is not known. Suppose, however, that we know that the intersection of the surface $z = f(x, y)$ and the plane $x = 1$ is given by the curve $\vec{r}_1(t) = \langle 1, 1 + t + t^2, 1 - 2t\rangle$, and the intersection of the surface $z = f(x, y)$ and the plane $y = 1$ is given by the curve $\vec{r}_2(t) = \langle t^3, 1, 2 - t\rangle$. We also know that $f(1, 1) = 1$. Find the partial derivatives $f_x(1, 1)$ and $f_y(1, 1)$.

18. The position of a particle at time t is given by

$$\vec{r}(t) = \langle -\frac{1}{2}t^2, 5t, \frac{1}{2}t^2 - 2t\rangle.$$

At what time is the speed a minimum?

19. Consider the plane P: $x - 2y + 2x = 5$, and the line L: $\vec{r}(t) = \langle 2t - 4, 2t + 1, t + 1\rangle$. Prove that P and L are parallel to each other.

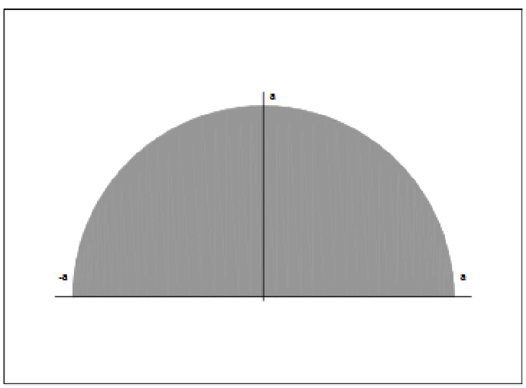
20. Find a parametric equation for the curve of intersection of the elliptical paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $x = y^2$.

21. In this problem, take $g = 10$ m/s^2. A projectile is fired at an angle α, $0 < \alpha < 90°$ on level ground with an initial speed of 120 m/s. Let $\vec{r}(t)$ be its position function of time.
    (a) Take $\vec{r}(0) = \langle 0, 0\rangle$, $\vec{r}'(0) = 120\langle \cos\alpha, \sin\alpha\rangle$, and $\vec{r}''(t) = \langle 0, -g\rangle$, and calculate $\vec{r}(t)$. Wind resistance is ignored.

    (b) Express the range of the projectile as a function of α.

    (c) If the range is 720 m, find two possible values of α. You may need the trig identity $\sin(2\alpha) = 2\sin\alpha\cos\alpha$.

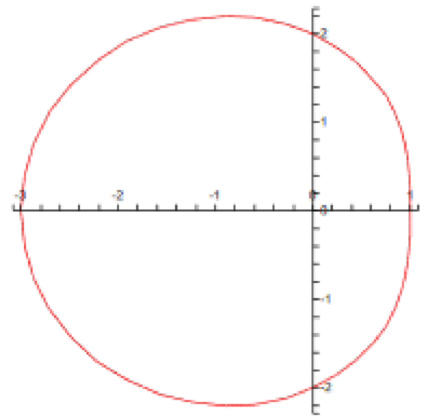
22. Find the mass of a non-homogenous lamina in the shape of a semi-circle of radius a (See the figure below) if the density function is $\delta(x, y) = y$ g/cm^2.



23. Find the amount (mass) of ice-cream in the ice-cream cone formed by a sphere of radius 7 cm centered at the origin and a cone opening upward from the origin with a top radius of 3 cm if the density of ice-cream is given by $\delta(x, y, z) = z$ g/cm^3.

24. The plot below depicts the curve whose equation in polar coordinates is

$$r = 2 - \cos\theta :$$

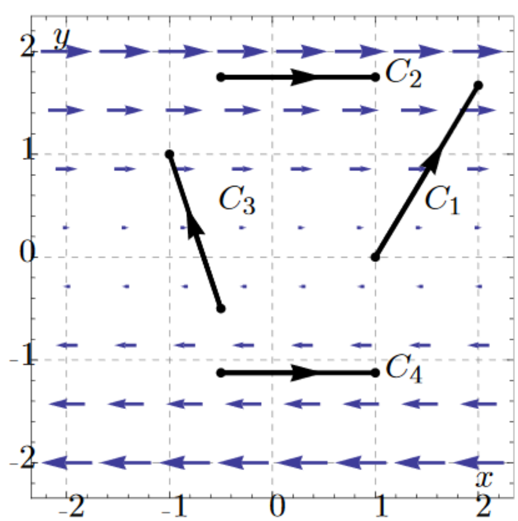


(a) Write an iterated double integral in polar coordinates whose numerical value equals the area enclosed by the curve.

(b) Evaluate your answer to (a).

25. Consider the disk $x^2 + y^2 \leq a^2$, where a is a positive constant. Set up a double integral that gives the average distance from any point (x, y) in this region to the origin. Evaluate your integral to find the average distance.

26. Consider the graph shown below, of the vector field $\vec{F}$ in the xy-plane and the line segments C_1, C_2, C_3 and C_4. The length of the vectors at each point is the strength of the vector field there.



(a) Place in order from smallest to largest, the six values $\int_{C_1} \vec{F} \cdot d\vec{r}$, $\int_{C_2} \vec{F} \cdot d\vec{r}$, $\int_{C_3} \vec{F} \cdot d\vec{r}$, $\int_{C_4} \vec{F} \cdot d\vec{r}$, $\int_{C_1} \vec{F} \cdot \hat{j}\, dy$, and the number 2.

(b) Is the vector field $\vec{F}$ a gradient field? Explain.

27. Consider the vector field $\vec{F}(x,y,z) = y\hat{\imath} + x\hat{\jmath} + \cos z\hat{k}$.
    (a) Show that this is a conservative vector field.

    (b) Let C be the skewed parabola $x(t) = t(2-t)$, $y(t) = t$ and $z(t) = t$ for $0 \le t \le 2$. Find $\int_C \vec{F} \cdot d\vec{r}$ in two different ways.

28. Prove that for the vector field $\vec{F}(x,y) = \langle 2y + \frac{3}{2}xy^2, 4x + \frac{3}{2}x^2y \rangle$ and the positively oriented curve C around any isosceles right triangle, $\oint_C \vec{F} \cdot d\vec{r} = a^2$, where a is the length of the legs of the triangle. (Note that you should not pick a specific triangle or value of a when computing the integral.)

29. (a) Evaluate

$$\int_E (x^2 + y^2 + z^2)^5 \, dV$$

    where E is the ball of radius 2 with center at the origin.

    (b) Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere centered at the origin with radius 3.

    (c) Find the gradient vector field of the function $f(x,y,z) = xe^{xyz}$.

    (d) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle x, z, y^2 \rangle$ and $\vec{r}(t) = \langle \sin t, t, t^3 \rangle$ for $0 \le t \le \pi/4$.

30. (a) Find the integral $\int_C (2x + y)\,dx + (x + 2y)\,dy$ where C is the curve $\vec{r}(t) = \langle \cos^3 t, e^{\sin t} \rangle$, with $0 \le t \le \pi$. [Hint: write the integral as $\int_C \vec{F} \cdot d\vec{r}$ and find f so that $\vec{\nabla} f = \vec{F}$.

    (b) Find $\int_C (2y - e^{\arctan x})\,dx + (x^3 + \cos(\sin y))\,dy$, where C is the boundary of the square $0 \le x \le 1$, $0 \le y \le 1$, oriented counterclockwise.

31. Consider the two-dimensional vector field shown below. Suppose you know that this is a conservative field on all of $\mathbb{R}^2$. Let C_1 be any path that starts at the origin and ends in quadrant I. Let C_2 be any path that starts at the origin and ends in quadrant IV. Let C_3 be the unit circle, oriented counterclockwise. List the following integrals in order from least to greatest:

$$\int_{C_1} \vec{F} \cdot d\vec{r}, \qquad \int_{C_2} \vec{F} \cdot d\vec{r}, \qquad \int_{C_3} \vec{F} \cdot d\vec{r}$$

32. Consider the lamina which lies entirely in the first quadrant and is bounded by the curves $y = \sqrt{9 - x^2}$ and $x = \sqrt{25 - y^2}$, as well as the x and y-axes. Suppose the mass of the lamina is given by

$$\delta(x,y) = \frac{xy}{x^2 + y^2}.$$

    Compute the mass of the lamina.

33. Consider the ice-cream cone and scoop bounded by the sphere of radius 5 cm, centered at the origin, and the cone centered around the positive z-axis and emanating from the origin such that it intersects the sphere in a circle of radius 3 cm. Suppose the cone is filled with ice cream at a density

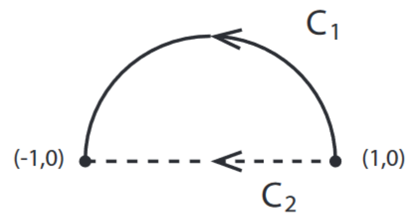
$$\delta(x,y,z) = z \text{ mg/cm}^3.$$

    (a) Use spherical coordinates to find the total mass of the cone.

    (b) Suppose I want to share this ice-cream cone with my little niece. Calculate the amount of ice-cream that is sticking out from the top of the cone. If I give the little girl the ice-cream that is sticking out from the top of the cone and keep the rest for myself, which one of us ends up with more delicious ice-cream? (Hint: use cylindrical coordinates).

34. Consider the two-dimensional force field $\vec{F}(x,y) = (4e^{-2x} + 3y^3)\hat{\imath} + 9xy^2\hat{\jmath}$.
  (a) Is $\vec{F}$ conservative? If so, find a potential function $f(x,y)$ whose gradient is $\vec{F}$.
  (b) Find the work done by the force field $\vec{F}$ in moving an object from $P = (0,1)$ to $Q = (1,2)$ via the path $y = x^2 + 1$.
  (c) Find the work done by the force field $\vec{F}$ in moving an object from $Q = (1,2)$ to $P = (0,1)$ via a straight line.

35. Let $\vec{F}(x,y) = \langle -y, x\rangle$.

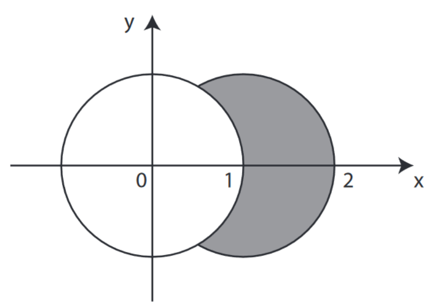


  (a) Let C_1 be the part of the unit circle $x^2 + y^2 = 1$ satisfying $y \geq 0$, oriented from $(1,0)$ to $(-1,0)$. Evaluate $\int_{C_1} \vec{F} \cdot d\vec{r}$.
  (b) Let C_2 be the line segment from $(1,0)$ to $(-1,0)$. Evaluate $\int_{C_2} \vec{F} \cdot d\vec{r}$.

36. Evaluate the integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{3(x^2+y^2)}}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

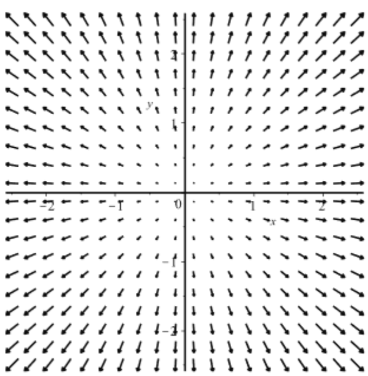
37. Find the area of the region inside the circle $x^2 + y^2 = 2x$ but outside of the circle $x^2 + y^2 = 1$. The region is shaded dark in the picture.



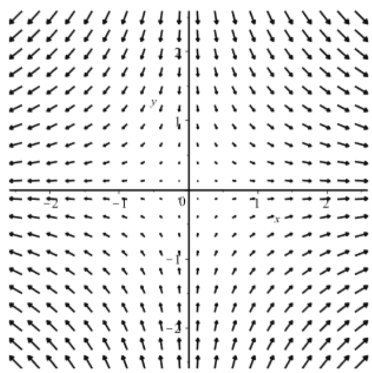
38. Let E be the solid which lies between the spheres of radius 1 and radius 2 centered at the origin and lies above the surface $z = \sqrt{x^2 + y^2}$. Evaluate

$$\int_E z\, dV$$

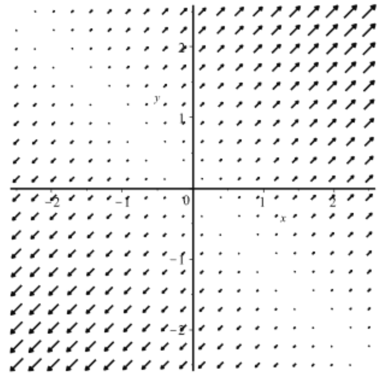
39. Consider the following two-dimensional vector fields in the xy-plane, and answer the questions below.



(a)



(b)



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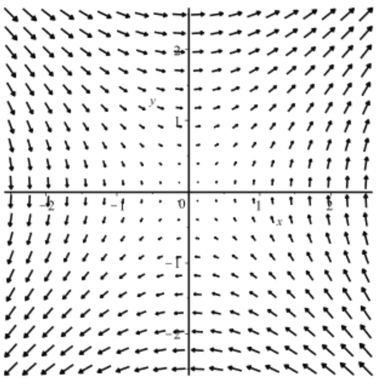

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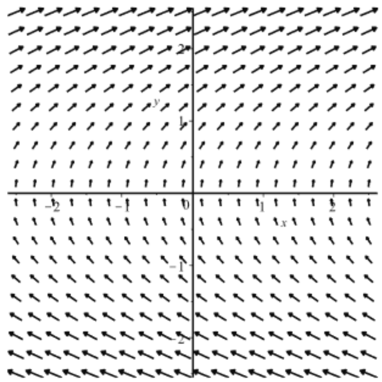

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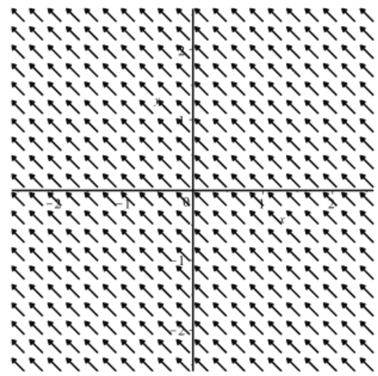

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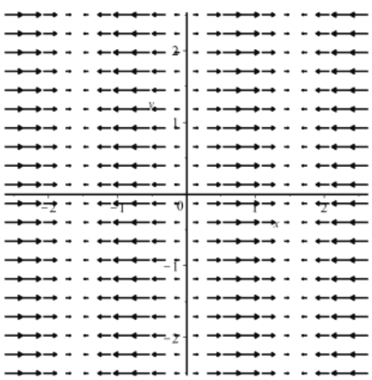

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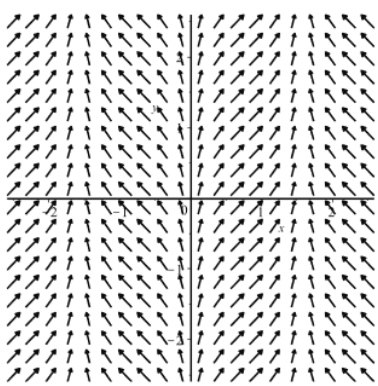

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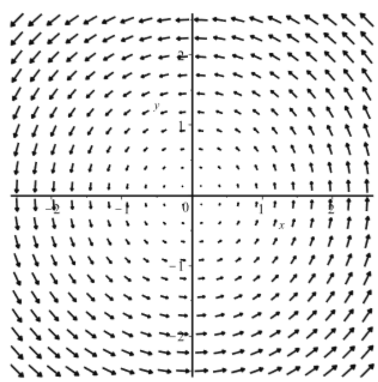

(i)

(i) Which picture represents the vector field $\langle -1, 1 \rangle$?

(ii) Which picture represents the vector field $\langle x, -y \rangle$?

(iii) Which picture represents the vector field $\langle -y, x \rangle$?

(iv) Which picture represents the gradient field of $f(x, y) = xy$?

40. Consider a ball of radius 1, $x^2 + y^2 + z^2 \leq 1$. Suppose that we use a cylindrical drill to bore a hole through the center of the ball. What should be the radius of the cylindrical drill so that the volume of the ring-shaped solid that remains has half the volume of the original ball? [This problem is challenging. Don't waste all of your time on it.]

41. The half cone $z = -\sqrt{x^2 + y^2}$ divides the ball $x^2 + y^2 + z^2 \leq 1$ into two parts. Evaluate the volume of the larger part.

42. A particle moves in a velocity vector field $\vec{v}(x, y) = \langle x^2, x + y^2 \rangle$. Suppose that the particle is at position $(2, 1)$ at time $t = 10$. Estimate the particle's position at time $t = 10.1$.