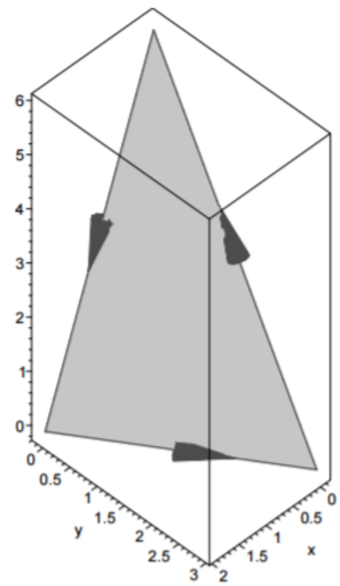
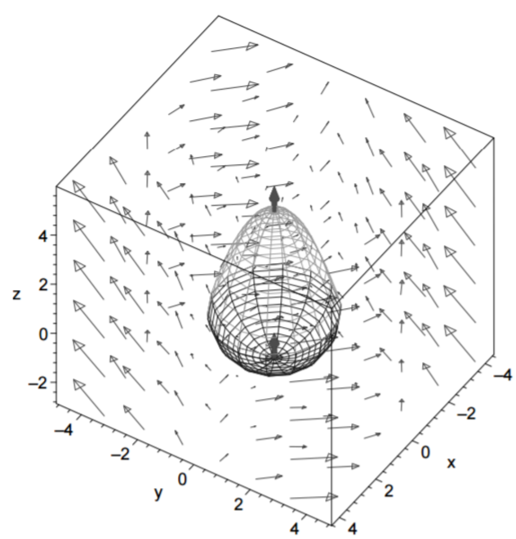
Exam 4 Study Guide

Math 223 Instructor: Dr. Gilbert

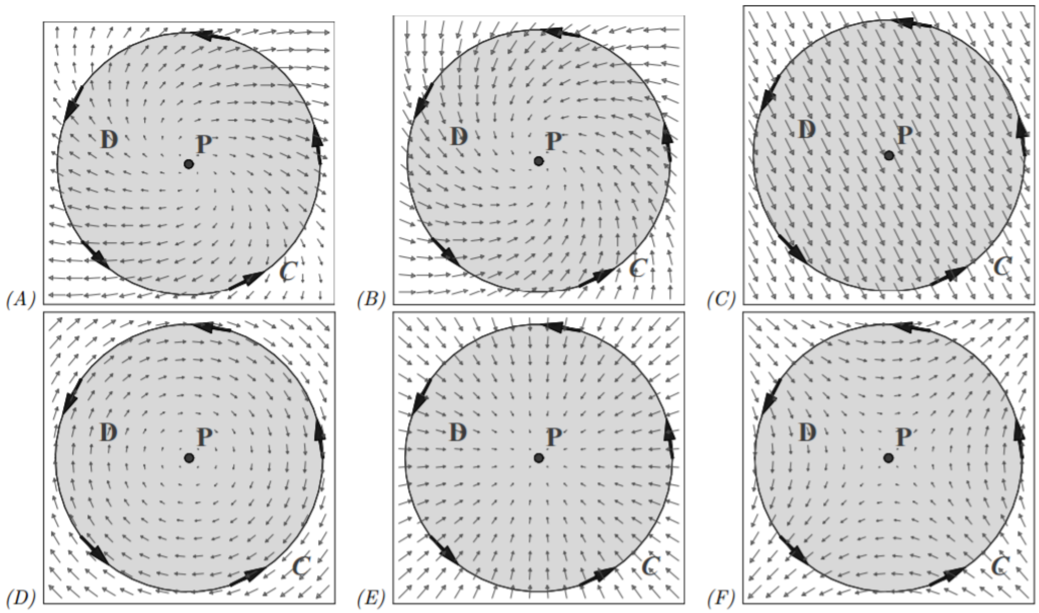
1. Find the circulation of the vector field $\vec{F}(x,y,z) = (4xy + xz)\hat{\imath} + (xy - yz)\hat{\jmath} + (z^2 - xz)\hat{k}$ along the curve which is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$, oriented as shown in the figure below.



2. Use the *Divergence Theorem* to compare (by evaluating the difference) the flux of the vector field $\vec{F}(x,y,z) = (-y^2 + \cos z)\hat{\imath} + (3xy + \sin z)\hat{\jmath} + (z + x^2)\hat{k}$ through the surface S_1 and S_2, where S_1 is the lower hemisphere of radius 2 centered at the origin, and S_2 is the part of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane and both surfaces are oriented upward (see the figure below).



3. Below are six pictures of vector fields $\vec{F}(x, y, z)$ shown in the xy-plane. As usual, the x-axis is horizontal and the y-axis is vertical. Assume that each are of the form $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$, where P and Q are functions of x and y alone (this does not mean that the vector fields do not live in $\mathbb{R}^3$). Also depicted in each picture below is a region D, its oriented boundary C, and a point P inside of D.



For each of the properties below, indicate all of the plots that have that property.

(a) curl $\vec{F}(P) \cdot \hat{k} > 0$

(b) Circulation of $\vec{F}$ around C is positive.

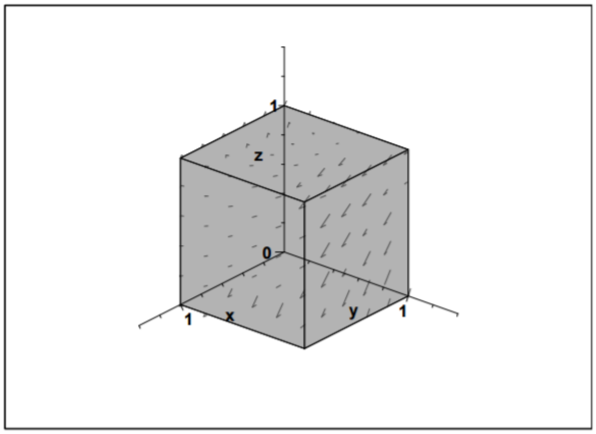
(c) Circulation of $\vec{F}$ around C is negative.

(d) div $\vec{F}(P) > 0$

(e) Flux of $\vec{F}$ across C is negative.

(f) $\vec{F}$ can be a gradient field.

4. Consider the vector field $\vec{F}(x,y,z) = (3x+2yz)\hat{\imath} + (2x-y+z)\hat{\jmath} + (x-3y+2z)\hat{k}$ and the unit cube in the first octant (see the figure below).

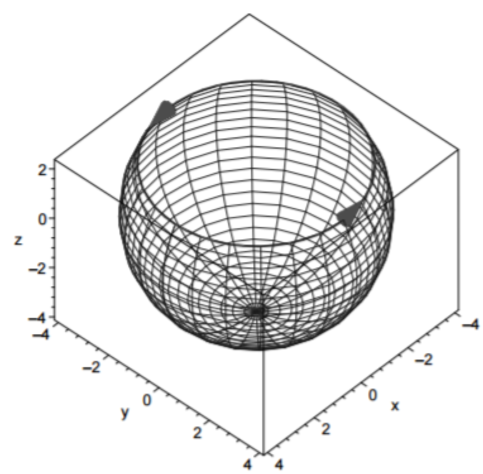


(a) Find the flux of $\vec{F}$ out of this cube.

(b) What is the flux of $\vec{F}$ through just the top surface of this cube (oriented upwards)?

5. Find the flux of the vector field $\vec{F}(x,y,z) = (x^2-y^2)\hat{\imath} + 2xz\hat{\jmath} + (z^2-2xz)\hat{k}$ across the surface S of the sphere of radius 3 centered at the origin.

6. Compute the circulation of the vector field $\vec{F}(x,y,z) = -y\hat{\imath} + (x-z)\hat{\jmath} + xz\hat{k}$ along the boundary of the part of the sphere of radius 4 (centered at the origin) below the plane $z=2$, if the boundary curve is oriented counterclockwise when viewed from above (see the figure below). Do the calculation in two different ways: directly and using *Stoke's Theorem.*



7. Consider the surface S which is the part of the plane $3x + y - z = 6$ contained inside the cylinder $x^2 + y^2 = 1$, oriented upwards.
   (a) Parametrize the boundary of S, $C = \partial S$. Make sure that C has the orientation inherited from S.
   (b) Find the flux of the vector field $\vec{F}(x, y, z) = xy\hat{\imath} + zy\hat{\jmath} - x^2\hat{k}$ across S.
   (c) Calculate the circulation $\oint \vec{F} \cdot d\vec{r}$ in two ways: using the parametrization you found in part (a) and then by utilizing *Stoke's Theorem.* If you get stuck on the algebra when you are directly computing the circulation, take a moment to appreciate Stoke's Theorem.

8. Let S be the portion of the surface $x = 5 - y^2 - z^2$ satisfying $x \geq 1$, oriented so that the unit normal vector at the point $(5, 0, 0)$ is $\hat{\imath}$. Let $\vec{F}(x, y, z) = \langle -1, 1, 0 \rangle$.
   (a) Set up and evaluate an iterated double integral equal to $\iint_S \vec{F} \cdot d\vec{A}$.
   (b) It turns out that $\vec{F} = \vec{\nabla} \times \vec{G}$, where $\vec{G} = \langle 0, z, -x \rangle$. Give an alternative calculation of the flux integral by applying Stoke's Theorem.

9. Suppose S is the lateral surface of the cylinder $x^2 + y^2 = 16$ satisfying $0 \leq z \leq 4$, oriented away from the origin.
   (a) Sketch S.
   (b) For the vector field $\vec{F} = \langle xe^z, ye^z, e^{xyz} \rangle$, compute $\iint_S \vec{F} \cdot d\vec{A}$.

10. Suppose that S is the orientable surface obtained by taking the union of the cylinder S_1 described by $x^2 + y^2 = 4$ with $0 \leq z \leq 4$, and the (upper) hemisphere, S_2, of radius 2 centered at $(0, 0, 4)$ (i.e. S_2 is the surface $x^2 + y^2 + (z - 4)^2 = 4$ satisfying $z \geq 4$). The orientation of $S = S_1 + S_2$ is away from the origin.
    (a) Sketch the surface S.
    (b) Compute
    $$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{A},$$
    where $\vec{F} = \langle yx^2 + \cos(zx), e^{xy} - xy^2, e^{xy} \rangle$.

11. Let E be the unit cube in the first octant (i.e. E has vertices $(0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$, and $(1, 0, 1)$). Let S denote the surface obtained by considering E without its top (i.e. the union of the five remaining sides of E). S is given the outward orientation. Compute
    $$\iint_S \vec{F} \cdot d\vec{A},$$
    where $\vec{F} = \langle x\cos(\pi y), -ye^z, yz \rangle$.

12. Suppose C is the curve parametrized by $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2 \rangle$ for $0 \leq t \leq 2\pi$. Compute the line integral of $\vec{F}(x, y, z) = \langle -x^2yz, xy^2z, e^{xy} \rangle$ along the curve C.

13. Let P be the prism bounded below by the plane $z = 0$, on the sides by the planes $y = 0$, $y = 5$ and $x = 0$, and bounded above by the plane $z = 2 - x$.
    (a) Sketch P.
    (b) Let S' be the boundary of P and let S be the surface obtained by removing the bottom face (i.e. the face in the xy-plane) from S'. We suppose that S is oriented outwards. Compute the flux across S of the vector field $\vec{F}(x, y, z) = \langle x^3y, x^2y^2, x^2y(z + 1) \rangle$.

14. Suppose C is the curve parametrized by $\vec{r}(t) = \langle 4\cos t, 4\sin t, 2\rangle$ for $0 \le t \le 2\pi$. Find

$$\oint_C \vec{F} \cdot d\vec{r},$$

where $\vec{F}(x,y,z) = \langle x^2 z, 2xz^2, \cos(xy)\rangle$.

15. Let W be the region in space bounded by the surfaces $z = -2$, $x^2 + y^2 = 4$, and $z = 4 - x^2 - y^2$.
(a) Sketch W.

(b) Let S' be the boundary of W and let S be the surface obtained by removing the bottom face (i.e. the disk in the plane $z = -2$) from S'. Suppose that S is oriented outwards. Find the flux across S of the vector field $\vec{F}(x,y,z) = \langle 2x^2 y, -2xy^2, (x^2+y^2)(z+1)\rangle$.

16. Suppose that S is the parabolic cap cut from the paraboloid $z = 15 - x^2 - y^2$ by the cone $z = 2\sqrt{x^2+y^2}$. That is, S is the part of the paraboloid $z = 15 - x^2 - y^2$ that lies above the cone $z = 2\sqrt{x^2+y^2}$. We orient S away from the origin.
(a) Sketch S.

(b) Suppose that $\vec{F}(x,y,z) = \langle x, y, z\rangle$. Find

$$\iint_S \vec{F} \cdot d\vec{A}$$

.

17. Suppose that C is the boundary of the triangle with vertices $(6,0,0)$, $(0,12,0)$, and $(0,0,3)$ with counterclockwise orientation when viewed from above. Suppose

$$\vec{F}(x,y,z) = \langle 2xz + \arctan(e^x), xz + (y+1)^y, \ln(1+z^2) + xy + y^2/2\rangle.$$

Compute $\oint_C \vec{F} \cdot d\vec{r}$.

18. Suppose a and b are real numbers with $0 \le a \le b$. Let W be the region in $\mathbb{R}^3$ that lies above the xy-plane, inside the sphere of radius b centered at the origin, and outside the cylinder of radius a centered about the z-axis.
(a) Sketch W.

(b) Let B be the bottom of W. That is, B is the surface in the xy-plane outside of the cylinder of radius a centered along the z-axis, and within the sphere of radius b centered at the origin. Sketch B.

(c) Let

$$\vec{F}(x,y,z) = \langle ye^{2y+\sin z}, xze^{-x\sin z}\cos x, (x^2+y^2+z^2)^{-1/2}\rangle.$$

Suppose B is oriented upward, and compute the flux of $\vec{F}$ across B.

(d) Let S' be the boundary of W, and let S be the surface obtained from S' by removing B. If S is oriented in the positive z-direction, then compute the flux of $\vec{F}$ across S, where $\vec{F}$ is as in part (c).

19. Suppose S is the surface that lies on the paraboloid $z = 9 - x^2 - y^2$ with $z \ge -7$.
(a) Sketch S.

(b) Let C be the boundary of S. Suppose that C is oriented counterclockwise when viewing the origin from the positive x-axis. Find $\oint_C \vec{F} \cdot d\vec{r}$ if $\vec{F}(x,y,z) = \langle -y^3 + e^x, x^3 + z + 4 - y^2, z + y + 7\rangle$.

20. Let E be the region in the first octant satisfying $0 \leq x \leq 1$, $0 \leq y \leq 2$, and $0 \leq z \leq 3$.
   (a) Sketch E.

   (b) Let S' be the boundary of E and let S be the surface obtained from S' by removing the bottom face (i.e. the surface in the xy-plane satisfying $0 \leq x \leq 1$ and $0 \leq y \leq 2$. We suppose that S is oriented outwards. Find the flux across S of the vector field $\vec{F}(x,y,z) = \langle \tan y + 3x^2y, e^z \sin x - 3zy^2x, (z-2)^2xy \rangle$.

21. Suppose that C is the curve parametrized by $\vec{r}(\theta) = \langle 5\cos\theta, 5\sin\theta, 7\rangle$ for $0 \leq \theta \leq 2\pi$.
   (a) Sketch C.

   (b) Compute the line integral of $\vec{F}(x,y,z) = \langle ze^{xz} - x^2y, xy^2, xe^{xz}\rangle$ along the curve C.

22. Let E be the prism bounded by the planes $z = -3$, $z = 5$, $x = 0$, $y = 0$, and $y = 6 - 2x$.
   (a) Sketch E.

   (b) Let P be the boundary of E and let S be the surface obtained by removing the top of P (i.e. the face in the plane $z = 5$. Suppose that both S and P have the outward orientation, and compute the flux across S of the vector field
$$\vec{F}(x,y,z) = \langle x^2y - xe^z, y - y^2x, e^z\rangle$$
   .

23. Let S be the surface described in cylindrical coordinates by the equation
$$z = r^2,$$
   where $0 \leq z \leq 4$, oriented in the negative z-direction.
   (a) Draw the surface S, along with a unit normal vector to show the orientation.

   (b) Describe ∂S, the boundary of S. Draw this curve in $\mathbb{R}^3$ with the orientation inherited from S.

   (c) Let $\vec{F}$ be the vector field
$$\vec{F}(x,y,z) = xz\hat{\imath} + yz^2\hat{\jmath} + (z+2)^{(z+1)^{xy}}\hat{k}.$$
   Evaluate $\displaystyle\iint_S \operatorname{curl}\vec{F}\cdot d\vec{A}$.

24. Suppose that the electric field in a charged plasma at some instant of time is given by $\vec{E}(x,y,z) = \langle 1-2x, 2+3y, 1+z\rangle$. According to Gauss's Law, the total electric charge contained within a closed surface S is proportional to the outward flux of the electric field across S. Let S be the surface whose sides S_1 are given by the piece of the paraboloid $x^2 + y^2 = z$ for $0 \leq z \leq 4$, and whose top S_2 is the disc of radius 2 which lies in the plane $z = 4$, centered at $(0,0,4)$.
   (a) Draw a picture of the surface S, and label S_1 and S_2 clearly.

   (b) Evaluate the outward flux of $\vec{E}$ across S.

25. Let $\vec{F} = \langle x, y, 0\rangle$, and let W be the solid region bounded by the surface $z = 1 - x^2 - y^2$ and the xy-plane. Let S be the boundary of W (both parts), oriented outward.
   (a) Calculate the flux of $\vec{F}$ across S by using the divergence theorem.

   (b) Recalculate the flux of $\vec{F}$ across S by direct calculation as the sum of two flux integrals.

26. Let S be the surface $z = 8 - 2x^2 - 2y^2$, $z \geq 0$, oriented in the positive z-direction. Let
$$\vec{F} = \langle x + y\cos z, 2xe^z - y, x^3y^2 - z^4 \rangle$$
Evaluate the integral $\displaystyle\iint_S \operatorname{curl} \vec{F} \cdot d\vec{A}$.

27. Let C be the curve of intersection of the plane $x - z = 2$ and the cylinder $x^2 + y^2 = 1$. The curve is oriented counterclockwise when viewed from above. Let
$$\vec{F}(x, y, z) = (-y + e^{-x^2})\hat{\imath} + x^2\hat{\jmath} - z^3\hat{k}.$$
Evaluate the circulation $\oint_C \vec{F} \cdot d\vec{r}$ of $\vec{F}$ along C.

28. Consider the vector field
$$\vec{F}(x, y, z) = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$
This vector field satisfies $\operatorname{div} \vec{F}(x, y, z) = 0$ for all (x, y, z) except $(0, 0, 0)$ at which the vector field is not defined. Let S be the closed surface given by the boundary of the box $-2 \leq x \leq 2$, $-3 \leq y \leq 3$, $-4 \leq z \leq 4$. We orient S outward. Evaluate the flux of $\vec{F}$ across S.

29. Let
$$\vec{F}(x, y, z) = (\cos z + xy^2)\hat{\imath} + xe^{-z}\hat{\jmath} + (\sin y + x^2 z)\hat{k}.$$
Let W be the solid region bounded by the surface $z = x^2 + y^2$ and the surface $z = 8 - x^2 - y^2$. Let S be the boundary of W, oriented away from W. Evaluate the flux of $\vec{F}$ across S.

30. Consider the surface $z = x^2 - y^2$. The cylinder $x^2 + y^2 = 1$ divides the surface into two parts, one of finite size and the other of infinite size. Let S be the part of finite size. In other words, S is the part of the surface $z = x^2 - y^2$ that lies inside of the cylinder $x^2 + y^2 = 1$. Assume that S is oriented upward and evaluate
$$\iint_S \langle x, 0, z \rangle \cdot d\vec{A}.$$

31. Evaluate the integral
$$\iint_S \langle x, y, 1 \rangle \cdot d\vec{A},$$
where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = x^2 + y^2$. Here the surface S is oriented upward (i.e. the $\hat{k}$-component of the unit normal vector is positive).

32. Consider the solid E that lies between the surfaces $z = \sqrt{x^2 + y^2}$, $x^2+y^2+z^2 = 4$, and $x^2+y^2+z^2 = 9$, and lies above the surface $z = \sqrt{x^2 + y^2}$. Let S be the boundary of the solid region E, oriented away from E. Let $\vec{F}(x, y, z)$ be the vector field
$$\vec{F}(x, y, z) = xy^2\hat{\imath} + yz^2\hat{\jmath} + zx^2\hat{k}.$$
Evaluate the flux integral
$$\iint_S \vec{F} \cdot d\vec{A}.$$

33. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$, $y \geq 0$, with orientation away from the origin. Evaluate the flux of the vector field
$$\vec{F}(x, y, z) = (x - 2yz)\hat{\imath} + (y + xz)\hat{\jmath} + (z + xy)\hat{k}$$
across S.