Investigating the relationship between restriction measures and self-avoiding walks

by

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As members of the Dissertation Committee, we certify that we have read the dissertation prepared by Michael James Gilbert entitled
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Dedication

For Julia. The light of my life.

Acknowledgments

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Abstract

It is widely believed that the scaling limit of the self-avoiding walk (SAW) is given by Schramm's $SLE_{8/3}$. In fact, it is known that if SAW has a scaling limit which is conformally invariant, then the distribution of such a scaling limit must be given by $SLE_{8/3}$. The purpose of this paper is to study the relationship between SAW and $SLE_{8/3}$, mainly through the use of restriction measures; conformally invariant measures that satisfy a certain restriction property.

Restriction measures are stochastic processes on randomly growing fractal subsets of the complex plane called restriction hulls, though it turns out that $SLE_{8/3}$ measure is also a restriction measure. Since SAW should converge to $SLE_{8/3}$ in the scaling limit, it is thought that many important properties of SAW might also hold for restriction measures, or at the very least, for $SLE_{8/3}$.

In [DGKLP2011], it was shown that if one conditions an infinite length self-avoiding walk in half-plane to have a bridge height at $y-1$, and then considers the walk up to height y, then one obtains the distribution of self-avoiding walk in the strip of height y. We show in this paper that a similar result holds for restriction measures $\mathbb{P}_\alpha$, with $\alpha \in [5/8, 1)$. That is, if one conditions a restriction hull to have a bridge point at some $z \in \mathbb{H}$, and considers the hull up until the time it reaches z, then the resulting hull is distributed according to a restriction measure in the strip of height $\mathrm{Im}(z)$. This relies on the fact that restriction hulls contain bridge points a.s. for $\alpha \in [5/8, 1)$, which was shown in [AC2010].

We then proceed to show that a more general form of that result holds for restriction hulls of the same range of parameters α. That is, if one conditions on the event that a restriction hull in $\mathbb{H}$ passes through a smooth curve γ at a single point, and then considers the hull up to the time that it reaches the point, then the resulting hull is distributed according to a restriction hull in the domain which lies underneath

the curve γ. We then show that a similar result holds in simply connected domains other than $\mathbb{H}$.

Next, we conjecture the existence of an object called the infinite length quarter-plane self-avoiding walk. This is a measure on infinite length self-avoiding walks, restricted to lie in the quarter plane. In fact, what we show is that the existence of such a measure depends only on the validity of a relation similar to Kesten's relation for irreducible bridges in the half-plane. The corresponding equation for irreducible bridges in the quarter plane, Conjecture 4.1.19, is believed to be true, and given this result, we show that a measure on infinite length quarter-plane self-avoiding walks analogous to the measure on infinite length half-plane self-avoiding walks (which was proven to exist in [LSW2002]) exists. We first show that, given Conjecture 4.1.19, the measure can be constructed through a concatenation of a sequence of irreducible quarter-plane bridges, and then we show that the distributional limit of the uniform measure on finite length quarter-plane SAWs exists, and agrees with the measure which we have constructed. It then follows as a consequence of the existence of such a measure, that quarter-plane bridges exist with probability 1.

As a follow up to the existence of the measure on infinite length quarter-plane SAWs, and the a.s. existence of quarter-plane bridge points, we then show that quarter plane bridge points exist for restriction hulls of parameter $\alpha \in [5/8, 3/4)$, and we calculate the Hausdorff measure of the set of all such bridge points.

Finally, we introduce a new type of (conjectured) scaling limit, which we are calling the fixed irreducible bridge ensemble, for self-avoiding walks, and we conjecture a relationship between the fixed irreducible bridge ensemble and chordal $SLE_{8/3}$ in the unit strip $\{z \in \mathbb{H} : 0 < \mathrm{Im}(z) < 1\}$.

Chapter 1
Introduction

1.1 Introduction

In recent years, much work has been done on discrete lattice models that arise in the study of statistical mechanics. These include the Ising model, critical percolation, loop-erased random walk, self-avoiding walks, etc. One of the most important problems posed throughout the study of these lattice models is the determination of a scaling limit. That is, a probability measure obtained as the lattice spacing goes to zero.

If the lattice model is two dimensional, we can identify the two-dimensional plane with the complex plane, and we can ask whether or not the scaling limit of such a lattice model is conformally invariant (invariant under conformal transformations), if such a scaling limit exists. The existence of such scaling limits had been conjectured by theoretical physicists for years, for many of the important lattice models that arise in the study of statistical mechanics, although rigorous mathematical proof of their existence remained elusive. However, in 2000, Oded Schramm introduced a stochastic process which was very successful in describing many of these scaling limits. It was originally referred to as *stochastic Lowener evolution*, but in years since has been referred to as the *Schramm-Lowener evolution*, or SLE_κ. It is technically defined as a family of conformal maps g_t which satisfy the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \qquad g_0(z) = z$$

where B_t is a standard one-dimensional Brownian motion. The random family of conformal maps give rise to an increasing family of hulls K_t such that g_t maps $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$. For $\kappa \leq 4$, these hulls are simple curves (and in fact it can be shown that the

Lowener chains are generated by curves for all $\kappa \leq 8$). This will be further discussed in Section 2.2.

Since its introduction, SLE_κ has been successfully used to describe the scaling limits of many important lattice models which arise in statistical mechanics, and as a result, it has been used to rigorously derive the values of many critical exponents. A few of these critical exponents had been calculated rigorously before, but the advent of SLE_κ allowed for a rigorous calculation of many more critical exponents.

One of the most important problems in the present-day study of statistical mechanics is the determination of the scaling limit of the self-avoiding walk, or SAW. It is conjectured that SAW has a scaling limit of $SLE_{8/3}$, and it has been shown [LSW2002] that if the scaling limit of self-avoiding walk exists, and if it is conformally invariant, then it must be given by $SLE_{8/3}$. In Section 2.2, we will describe in detail what it means for SAW to have a scaling limit, and what it means for this scaling limit to be conformally invariant.

The purpose of this paper is to study the relationship between SAW and $SLE_{8/3}$ through the use of *restriction measures*. This is a stochastic process on randomly growing families of restriction hulls (see Section 2.3) which are conformally invariant, and which satisfy the restriction property, which will also be discussed in Section 2.3.

In Chapter 2, we will provide all the requisite background information required for the remainder of the paper. In 2.1, we describe half-plane self-avoiding walks and even briefly discuss the construction of the infinite upper half-plane SAW. In Section 2.2, we will discuss many of the important results from complex analysis which will be used throughout the paper. We will also describe scaling limits, conformal invariance, and precisely state the conjecture that SAW converges to $SLE_{8/3}$ in the scaling limit. In 2.3, we will review many important facts about restriction measures, as well as briefly discuss the construction of such measures.

In Chapter 3, we will show that if we consider restriction hulls K on the triple $(\mathbb{H}, 0, \infty)$, and condition on the event that K has a bridge point at $z \in \mathbb{H}$ and consider the hull K up until the first time it touches z, then the resulting hull is distributed according to a restriction measure with the same parameter, on the domain $\{w \in \mathbb{H} : 0 < \mathrm{Im}(w) < \mathrm{Im}(z)\}$, from 0 to z. We then proceed to prove a generalization of this theorem, which requires a kind of *generalized bridge point*. We define a generalized bridge point to be a point where a restriction hull K on the triple $(\mathbb{H}, 0, \infty)$, intersects a smooth curve $\gamma : [a, b] \to \mathbb{H}$, where $\gamma(a, b) \subset \mathbb{H}$, at a single point. We show then, that it follows from conformal invariance that the same type of result holds in arbitrary simply connected domains D other than all of $\mathbb{C}$.

In Chapter 4, we construct an object called the infinite length quarter-plane self-avoiding walk. We define a type of SAW called a *quarter-plane bridge*, and we essentially construct the measure by defining a measure on irreducible quarter plane bridges and then concatenating an i.i.d. sequence of such bridges. One might wonder, however, if there aren't more natural measures on infinite quarter plane SAWs. By it's construction, the previously mentioned measure is supported on concatenated sequences of i.i.d. irreducible bridges, so what it really gives us is a measure on infinite length quarter-plane bridges. For this reason, we prove the existence of the distributional limit on the uniform measure of n-step quarter-plane SAWs as $n \to \infty$, and we show that it coincides with the measure we have constructed.

The construction of the infinite length quarter-plane SAW was then motivation to consider restriction measures defined in the quarter-plane. In Chapter 5, we prove the existence of quarter-plane bridge points for restriction hulls K under the law of restriction measures in the quarter-plane, starting at 0 and ending at ∞. In fact, we show more. If one considers a restriction measure with parameter $\alpha > 0$, then we show that with probability 1, the Hausdorff measure of the set of quarter-plane bridge points of a given hull K under the law of the restriction measure, is $\min(0, 2 - 8/3\alpha)$. This is interesting because it is known that for restriction measures in the half-plane, bridge points exist with probability 1, and the Hausdorff dimension of the set of bridge points is given by $\min(0, 2 - 2\alpha)$. Thus, in the half-plane, bridge points exist for all $\alpha \in [5/8, 1)$. However, in the quarter-plane, quarter-plane bridge points cease to exist for all $\alpha > 3/4$. The $\alpha = 3/4$ case remains unknown.

Finally, in Chapter 6, we introduce a new ensemble for self-avoiding walks as follows: take a self-avoiding walk of infinite length in the half-plane distributed according to the measure constructed in [LSW2002], [DGKLP2011], consider it up to the n-th bridge height and scale by the reciprocal of the n-th bridge height to obtain a curve in the unit strip $\{z \in \mathbb{H} : 0 < \text{Im}(z) < 1\}$. These curves inherit a distribution from the measure on the original SAW, and if one takes the limit as $n \to \infty$, one obtains an ensemble of curves spanning the unit strip, ending anywhere along the upper boundary of the unit strip. A natural question to ask would be whether or not these curves are distributed according to $SLE_{8/3}$, starting at 0 and ending at $x + i$, integrated along the conjectured exit density of the scaling limit of SAW in the unit strip starting at 0 and ending anywhere along the upper boundary. We argue that this is not the case, but that one can obtain the $SLE_{8/3}$ distribution integrated along such an exit density if one first weights each of the scaled walks in the unit strip by the n-th bridge height raised to an appropriate power (before taking the limit $n \to \infty$). In addition to a heuristic argument in support of this, we provide numerical evidence in support of the conjecture, and this allows us to give an estimate on the boundary scaling exponent for self-avoiding walk, which agrees with the conjectured value for the boundary scaling exponent within an error of 0.000303.

1.2 Remark on notations

Throughout this paper we use the standard convention of letting $\mathbb{C}$ denote the complex plane and $\mathbb{R}$ the set of real numbers. The set of natural numbers will be denoted by $\mathbb{N}$, while the set of integers will be $\mathbb{Z}$. For $z \in \mathbb{C}$, we let Re z and Im z denote the real and imaginary parts of z, respectively. $|z| = \sqrt{(\text{Re } z)^2 + (\text{Im } z)^2}$ is referred to as the *modulus*, or *absolute value*, of z. Given a set A, we let $\overline{A}$ denote the closure of

A, A° denote the interior of A and ∂A the boundary of A.

We will denote the open unit disk in $\mathbb{C}$ by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the upper half plane by $\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$. Given $d \in \mathbb{N}$, we let $\mathbb{Z}^d$ be the set of points in $\mathbb{R}^d$ whose coordinates are integers. Furthermore, in the case $d = 2$, we consider $\mathbb{Z}^2$ as $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$, the set of points in the complex plane with real and imaginary parts in $\mathbb{Z}$.

Suppose that $f(x)$ and $g(x)$ are functions defined on some subset of $\mathbb{R}$. We will use the following notations when referring to asymptotic results concerning f and g in the limit as $x \to a$ for $a \in [-\infty, +\infty]$.

- $f(x) \sim g(x)$ if $\lim_{x\to a} f(x)/g(x) = 1$.
- $f(x) \approx g(x)$ if $\log f(x) \sim \log g(x)$.
- $f(x) \asymp g(x)$ if there exist positive constants c_1 and c_2 such that

  $$c_1 f(x) \le g(x) \le c_2 f(x)$$

  for all x sufficiently close to a.

Chapter 2
Background

In an effort to keep this paper relatively self contained, this chapter is dedicated to providing the requisite background information necessary for the results found in subsequent chapters. This chapter contains no new results, and the information contained in it can be found in [MS1993],[DGKLP2011],[LSW2002],[LSW2003],[Lawler2008]. In section 2.1 we will review the self-avoiding walk, including the construction of the infinite half-plane self-avoiding walk and the bridge decomposition thereof. In section 2.2, we introduce the notion of conformal invariance and the Schramm-Loewner evolution, as well as reviewing some essential facts from complex analysis. In section 2.3, we introduce restriction measures and briefly review their construction.

2.1 The half-plane self-avoiding walk in $\mathbb{Z}^2$

Throughout this paper we will primarily consider self-avoiding walks on the lattice $\delta\mathbb{Z}^2 = \delta\mathbb{Z} + i\delta\mathbb{Z}$ for $\delta > 0$. In this section we fix $\delta = 1$ and discuss results for self-avoiding walks on $\mathbb{Z}^2$. Most of the results we mention hold for self-avoiding walks on the lattice $\mathbb{Z}^d$ for any dimension $d \geq 2$, and much of it can be found in [MS1993], [DGKLP2011], [LSW2002].

2.1.1 Full-plane SAW

Definition 2.1.1. An N-step *self-avoiding walk* (SAW) on the lattice $\mathbb{Z}^2$ beginning at $x \in \mathbb{Z}^2$ is a sequence of lattice sites $\omega = [\omega_0, \dots, \omega_N]$ which satisfy the following:

- $|\omega_j - \omega_{j-1}| = 1$ for all $j = 1, \dots, N$
- $\omega_j \neq \omega_k$ for $j \neq k$

- $\omega_0 = x$.

We can realize $\mathbb{Z}^2$ as the set of all complex points z whose real and imaginary parts are integers, along with the line segments connecting neighboring points, which we call *nearest neighbor bonds*. In doing so, we may realize a given N-step SAW as a simple curve in $\mathbb{C}$ by connecting consecutive points in the sequence through the corresponding nearest neighbor bond.

Let $\mathcal{S}_N$ denote the set of all N-step SAWs in $\mathbb{Z}^2$ beginning at the origin. Let $c_N := |\mathcal{S}_N|$ denote the cardinality of $\mathcal{S}_N$. By convention, we take $c_0 = 1$ (i.e. the trivial walk $\omega = 0$). We realize $\mathcal{S}_N$ as a probability space by equipping it with the uniform measure, $\mathbf{P}_N$. That is, given $\omega \in \mathcal{S}_N$, we define $\mathbf{P}_N(\omega) = 1/c_N$.

Although it is difficult to determine c_N for large values of N, one might hope that it is possible to determine some asymptotic results concerning c_N as $N \to \infty$. It is conjectured that there exist lattice-independent *critical exponents* ν and γ such that

$$c_N \sim A\beta^N N^{\gamma-1} \tag{2.1.2}$$

for some positive constant A, and

$$\mathbf{E}_N[|\omega_N|^2] \sim CN^{2\nu}, \tag{2.1.3}$$

where $\mathbf{E}_N$ denotes expectation with respect to $\mathbf{P}_N$ and C is a positive constant. The constant β in (2.1.2) is referred to as the *connective constant*, and is lattice-dependent. Both equations (2.1.2) and (2.1.3) remain conjecture, though it is not very difficult to show a weaker form of (2.1.2), namely

$$c_N \approx \beta^N. \tag{2.1.4}$$

This is most easily seen through the process of *concatenation*.

Definition 2.1.5. Suppose $\omega^1 \in \mathcal{S}_N$ and $\omega^2 \in \mathcal{S}_M$. The concatenation of ω^1 with ω^2, denoted $\omega^1 \oplus \omega^2$, is the $N + M$ step SAW beginning at 0 defined by

$$\omega^1 \oplus \omega^2_j = \begin{cases} \omega^1_j, & j = 0, \dots, N \\ \omega^1_N + \omega^2_{j-N}, & j = N+1, \dots, N+M \end{cases}$$

Notice that every SAW in $\mathcal{S}_{N+M}$ can be written as the concatenation of a SAW in $\mathcal{S}_N$ with a SAW in $\mathcal{S}_M$, though not every concatenation of a SAW in $\mathcal{S}_N$ with a SAW in $\mathcal{S}_M$ is self-avoiding. Since $c_N c_M$ is the number of concatenations of walks in $\mathcal{S}_N$ with walks in $\mathcal{S}_M$, we thus have

$$c_{N+M} \leq c_N c_M. \tag{2.1.6}$$

Therefore, we see that the sequence $(\log c_N)$ is subadditive, and by Proposition A.1.1, the limit

$$\log \beta := \lim_{N\to\infty} \frac{\log c_N}{N} \tag{2.1.7}$$

exists. We refer to the constant β in equation (2.1.7) as the connective constant, and (2.1.4) follows.

2.1.2 Half-plane SAW

In this section we will review the construction of the infinite half-plane SAW, a measure on SAWs of infinite length which stay in the upper half-plane $\mathbb{H}$. We begin by first defining SAWs in the half-plane of finite length. A *half-plane self-avoiding walk* starting at 0 is defined to be a SAW which stays in the upper half plane $\mathbb{H}$. Formally,

Definition 2.1.8. An N-step half-plane self-avoiding walk beginning at 0 is defined to be an $\omega \in \mathcal{S}_N$ which satisfies

$$\mathrm{Im}(\omega_j) > 0 \tag{2.1.9}$$

for all $j = 1, \ldots, N$. Let $\mathcal{H}_N$ denote the set of all $\omega \in \mathcal{S}_N$ satisfying (2.1.9) and $h_N := |\mathcal{H}_N|$. By convention, we take $h_0 = 1$.

Perhaps the most important step in the construction of the infinite half-plane SAW is the introduction of a *bridge*.

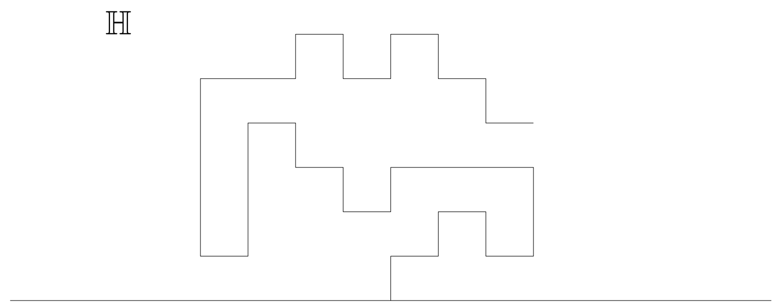

FIGURE 2.1. A 37-step half-plane SAW.

Definition 2.1.10. An n-step bridge is an n-step self-avoiding walk, ω, the imaginary parts of which satisfy

$$\mathrm{Im}(\omega_0) < \mathrm{Im}(\omega_j) \le \mathrm{Im}(\omega_n). \tag{2.1.11}$$

We will let $\mathcal{B}_n$ denote the set of n-step bridges beginning at 0 and $b_n := |\mathcal{B}_n|$. By convention, we take $b_0 = 1$.

It is clear that every $\omega \in \mathcal{B}_n$ is also in $\mathcal{H}_n$. Furthermore, if $\omega^1 \in \mathcal{B}_n$ and $\omega^2 \in \mathcal{B}_m$, then we have $\omega^1 \oplus \omega^2 \in \mathcal{B}_{n+m}$, and it follows that

$$b_n b_m \le b_{n+m}. \tag{2.1.12}$$

It follows that the sequence $(-\log b_n)$ is subadditive, and thus, once again by Proposition A.1.1, the limit

$$\beta_{bridge} = \lim_{n\to\infty} b_n^{1/n} \tag{2.1.13}$$

exists and is equal to $\sup_{n\ge 1} b_n^{1/n}$. Since $b_n \le c_n$ for all $n \ge 1$, we have $\beta_{bridge} \le \beta$. It is known, however, that indeed, $\beta_{bridge} = \beta$. For a proof of this fact, the reader is referred to [MS1993]

We have seen that the concatenation of two bridges always gives rise to another bridge. However, it is not true that every bridge can be written as the concatenation of two (non-trivial) bridges. In the latter case, we call the bridge an *irreducible bridge*.

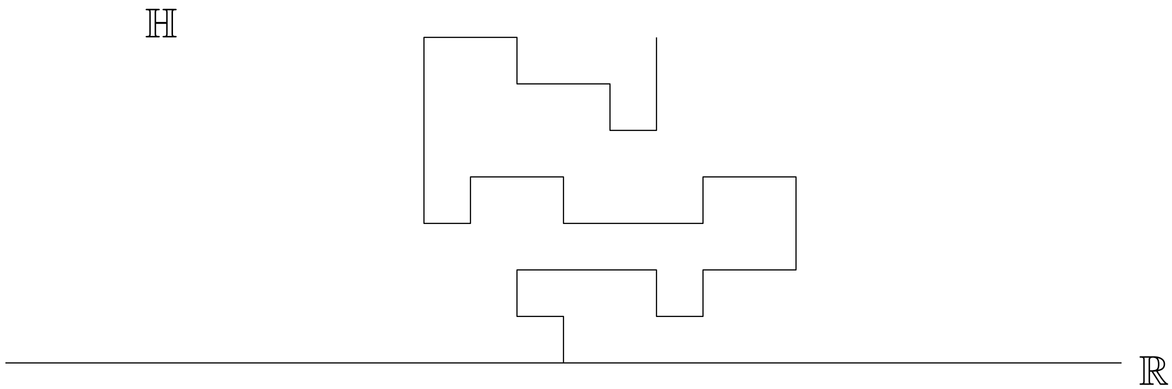

FIGURE 2.2. A 37-step bridge.

We will see that irreducible bridges turn out to be the building blocks of the infinite half-plane SAW.

Let $\mathcal{I}_n$ denote the set of all n-step irreducible bridges beginning at the origin, and set $\lambda_n := |\mathcal{I}_n|$. By convention, we take $\lambda_0 = 0$. It will be useful to consider sets of walks of variable length. Let $\mathcal{H} = \bigcup_{N=0}^{\infty} \mathcal{H}_N$ be the set of all half-plane SAWs of any length, and similarly let $\mathcal{S} = \bigcup_{N=0}^{\infty} \mathcal{S}_N$, $\mathcal{B} = \bigcup_{n=0}^{\infty} \mathcal{B}_n$ and $\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n$. We will make use of the following *generating functions*:

Definition 2.1.14. The generating functions for the sequences (c_N), (h_N), (b_n) and (λ_n) are defined by the formulae

$$S(z) = \sum_{\omega \in \mathcal{S}} z^{|\omega|} = \sum_{N=0}^{\infty} c_N z^N$$

$$H(z) = \sum_{\omega \in \mathcal{H}} z^{|\omega|} = \sum_{N=0}^{\infty} h_N z^N$$

$$B(z) = \sum_{\omega \in \mathcal{B}} z^{|\omega|} = \sum_{n=0}^{\infty} b_n z^n$$

$$I(z) = \sum_{\omega \in \mathcal{I}} z^{|\omega|} = \sum_{n=1}^{\infty} \lambda_n z^n,$$

where $|\omega|$ denotes the *length* of ω, or the number of steps of ω.

The first thing to observe is that by definition, $z_c := \beta^{-1}$ is the radius of convergence for the series $S(z)$. Also, according to (A.1.2), we have

$$\beta^N \leq c_N \text{ for all } N \geq 1. \tag{2.1.15}$$

This allows us to prove the following form of "continuity" of $S(z)$ at z_c:

Proposition 2.1.16.

$$S(z_c) := \lim_{z \to z_c-} S(z) = +\infty. \tag{2.1.17}$$

Proof. (2.1.15) gives us, for $z < z_c$,

$$\begin{aligned} S(z) &= \sum_{N=0}^{\infty} c_N z^N \\ &\geq \sum_{N=0}^{\infty} (\beta z)^N \\ &= \frac{1}{1 - \beta z}, \end{aligned}$$

which tends to $+\infty$ as z tends to z_c from below. □

The construction of the infinite half-plane SAW relies on the following Proposition, originally due to Kesten [Kesten1963]. The proof requires the notion of the span of a self-avoiding walk, which we will define now.

Definition 2.1.18. The *span* of a self-avoiding walk $\omega \in \mathcal{S}_N$ is defined by

$$\operatorname{span}(\omega) = \max_{1 \leq j \leq N} \operatorname{Im}(\omega_j) - \min_{1 \leq j \leq N} \operatorname{Im}(\omega_j). \tag{2.1.19}$$

The number of N-step SAWs beginning at the origin with span A will be denoted by $c_{N,A}$. We similarly define $h_{N,A}$, $b_{N,A}$, etc.

Proposition 2.1.20. $B(z_c) = \infty$ *and hence* $I(z_c) = 1$.

Before proceeding to the proof of Proposition 2.1.20, we first prove a useful lemma.

Lemma 2.1.21. *If $\omega \in \mathcal{H}_N$, then ω can be written as $\omega = \omega^1 \oplus (-\omega^2) \oplus \cdots \oplus (-1)^{k-1}\omega^K$, where $\omega^k \in \mathcal{B}$ for all $k = 1, \ldots, K$. Furthermore, if $A_k = \mathrm{span}(\omega^k)$, then we have $A_1 > A_2 > \cdots > A_K > 0$.*

Proof. Let $\omega \in \mathcal{H}_N$. Define $A_1 = A_1(\omega)$ to be the maximum value that the imaginary part of ω takes on. That is, $A_1 = \max_{1\le j\le N} \mathrm{Im}(\omega_j)$.Then define $n_1 = n_1(\omega)$ to be the last j, $j = 1, \ldots, N$ such that $\mathrm{Im}(\omega_j) = A_1$. Then, recursively define A_k by

$$A_k = \max_{n_{k-1}\le j\le N} (-1)^{k-1} \left(\mathrm{Im}(\omega_j) - \mathrm{Im}(\omega_{n_{k-1}})\right),$$

and n_k is the last time that the imaginary part of ω reaches A_k. Then the decomposition is obtained by taking $\omega^1 = [\omega_0, \omega_1, \ldots, \omega_{n_1}]$, and in general by taking $\omega^k = [\omega_{n_{k-1}}, \ldots, \omega_{n_k}]$ for $k = 1, \ldots, K$. □

Proof of Proposition 2.1.20. The proof here follows what can be found in [MS1993] and [DGKLP2011]. To begin, notice that every $\omega \in \mathcal{B}$ can be written uniquely as $\omega^1 \oplus \omega^2$, where $\omega^1 \in \mathcal{I}$ and $\omega^2 \in \mathcal{B}$. This leads us to

$$b_n = \delta_{0,n} + \sum_{m=1}^{n} \lambda_m b_{n-m}, \tag{2.1.22}$$

for all n. From (2.1.22) we have

$$B(z) = 1 + I(z)B(z),$$

from which we immediately conclude that

$$B(z) = \frac{1}{1 - I(z)}. \tag{2.1.23}$$

Thus, if we can show that $B(z_c) = \infty$, the proof of the Proposition will be complete.

Given $\omega \in \mathcal{B}$, let $h(\omega)$ denote the height of the bridge. That is, $h(\omega) = \max_k\{\mathrm{Im}(\omega_k)\}$. Also, note that given $\omega \in \mathcal{S}$, if j is the largest integer less than or equal to $|\omega|$ such that $\omega_j = \min_k\{\mathrm{Im}(\omega_k)\}$, then $\omega^1 = [\omega_j, \omega_{j-1}, \ldots, \omega_0]$ and $\omega^2 = [\omega_j, \omega_{j+1}, \ldots, \omega_N]$ are both half-plane SAWs. Here $|\omega| = N$ is the length of ω. This implies that

$$c_n \le \sum_{m=0}^{\infty} h_m h_{n-m} \tag{2.1.24}$$

Now, by Lemma 2.1.21, every $\omega \in \mathcal{H}_N$ can be decomposed into a sequence of K bridges, each with m_k number of steps, such that $\sum_k m_k = N$. Furthermore, if the bridge with length m_k in the decomposition has span A_k, then we have $A_1 > A_2 > \cdots > A_K > 0$. Since this transformation is one-to-one, we have

$$h_N \leq \sum \left(\prod_{k=1}^{K} b_{m_k, A_k} \right), \tag{2.1.25}$$

where the sum is over all positive integers K, all sequences of positive numbers $A_1, \ldots, A_K$ such that $A_1 > A_2 > \cdots > A_K > 0$ and all integers $m_k \geq 1$ such that $\sum_{k=1}^{K} m_k = N$. Therefore, we can see that

$$\sum_{N=0}^{\infty} h_N z^N \leq \prod_{A=1}^{\infty} \left(1 + \sum_{m=1}^{\infty} b_{m,A} z^m \right),$$

which can be seen by comparing z^N terms on both sides of the inequality and using (2.1.25). Since $1 + x \leq e^x$ for all x, this leads to

$$\begin{aligned} H(z) &\leq \exp \left(\sum_{A=1}^{\infty} \sum_{m=1}^{\infty} b_{m,A} z^m \right) & (2.1.26) \\ &= e^{(B(z)-1)}. & (2.1.27) \end{aligned}$$

By (2.1.24), we can see that $S(z_c) = +\infty$ implies that $H(z_c) = +\infty$, and so consequently, by (2.1.27), $B(z_c) = +\infty$. The result that $I(z_c) = 1$ follows simply now from (2.1.23). $\square$

From here on, given an integer $j > 0$, we will identify $(\omega^1, \ldots, \omega^j) \in \mathcal{I}^j$ with the concatenation $\omega^1 \oplus \cdots \oplus \omega^j$. This is a one-to-one correspondence, so the identification is well-defined. Then $\mathcal{I}^\infty = \mathcal{I} \times \mathcal{I} \times \cdots$ is the set of all concatenations of infinitely many irreducible bridges beginning at 0. We will let $\mathcal{H}_\infty$ denote the set of all infinite length upper half-plane SAWs beginning at 0. Given $\omega^1 \oplus \cdots \oplus \omega^j \in \mathcal{I}^j$, we will let $\mathcal{H}_\infty(\omega^1, \ldots, \omega^j)$ denote the "cylinder" set of all $\omega \in \mathcal{H}_\infty$ such that $\omega = \omega^1 \oplus \cdots \oplus \omega^j \oplus \tilde{\omega}$, where $\tilde{\omega} \in \mathcal{H}_\infty$. We will define the infinite upper half-plane SAW as follows:

- Let $\mu_{\mathcal{I}}$ be the measure on $\mathcal{I}$ such that $\mu_{\mathcal{I}}(\omega) = \beta^{-|\omega|}$ for $\omega \in \mathcal{I}$.

- Let $\mu_{\mathcal{I}^j}$ be the measure on $\mathcal{I}^j$ defined by product measure, so $\mu_{\mathcal{I}^j}(\omega^1 \oplus \cdots \oplus \omega^j) = \mu_{\mathcal{I}}(\omega^1) \cdots \mu_{\mathcal{I}}(\omega^j)$. We will also write $\mu_{\mathcal{I}^j}$ for the extension of $\mu_{\mathcal{I}^j}$ to $\mathcal{H}$ with $\mu_{\mathcal{I}^j}(\mathcal{H} \setminus \mathcal{I}^j) = 0$. Here we are setting $\mu_{\mathcal{I}^j}(\omega) = 0$ if ω cannot be written as $\omega = \omega^1 \oplus \cdots \oplus \omega^j$ with $\omega^1, \ldots, \omega^j \in \mathcal{I}$.

- We define $\mu_{\mathcal{I}^\infty}$ on $\mathcal{I}^\infty$ by extension. That is, we extend the measure $\mu_{\mathcal{I}}$ to $\mathcal{H}_\infty$ by defining $\mu_{\mathcal{I}}(\mathcal{H}_\infty(\omega)) = \mu_{\mathcal{I}}(\omega)$, and similarly with $\mu_{\mathcal{I}^j}$. If $\omega \in \mathcal{H}_\infty$ cannot be written as $\tilde{\omega} \oplus \hat{\omega}$ for $\tilde{\omega} \in \mathcal{I}$ and $\hat{\omega} \in \mathcal{H}_\infty$, we define $\mu_{\mathcal{I}}(\omega) = 0$. We then define $\mu_{\mathcal{I}^\infty}$ by the Kolmogorov extension Theorem. If we write $\mathbf{P}_{\mathbb{H},\infty}$ in place of $\mu_{\mathcal{I}^\infty}$, then $\mathbf{P}_{\mathbb{H},\infty}(\mathcal{H}_\infty \setminus \mathcal{I}^\infty) = 0$, and according to this definition, we have $\mathbf{P}_{\mathbb{H},\infty}(\mathcal{H}_\infty(\omega^1, \ldots, \omega^j)) = \beta^{-m}$ for $\omega^1, \ldots, \omega^j \in \mathcal{I}$, $|\omega^1| + \cdots + |\omega^j| = m$.

By *Kesten's relation*,

$$\sum_{\omega \in \mathcal{I}} \beta^{-|\omega|} = 1. \tag{2.1.28}$$

This was proven in Proposition 2.1.20, and shows that $\mathbf{P}_{\mathbb{H},\infty}$ defines a probability measure on infinite length SAWs in $\mathbb{H}$ beginning at 0. We take this to be the definition of the infinite upper half-plane SAW. In [MS1993], this measure was referred to as the *infinite bridge measure*, and this is perhaps a more intuitive name for $\mathbf{P}_{\mathbb{H},\infty}$. However, it has been shown that this measure is equivalent to other measures which are, perhaps, more aptly referred to as the infinite upper half-plane self-avoiding walk.

For example, Lawler, Schramm and Werner showed in [LSW2002] that the weak limit as $N \to \infty$ on the uniform measures, $\mathbf{P}_{\mathbb{H},N}$, on N-step upper half-plane SAWs exists and gives a measure on $\mathcal{H}_\infty$. In particular, if we let $\mathcal{H}(\omega^1, \ldots, \omega^j)$ denote the set of $\omega \in \mathcal{H}$ such that $\omega = \omega^1 \oplus \cdots \oplus \omega^j \oplus \tilde{\omega}$, $\omega^1, \ldots, \omega^j \in \mathcal{I}$, $\tilde{\omega} \in \mathcal{H}$, $|\omega^1| + \cdots + |\omega^j| = m$, then [LSW2002] shows that

$$\lim_{N \to \infty} \mathbf{P}_{\mathbb{H},N}(\mathcal{H}(\omega^1, \ldots, \omega^j)) = \beta^{-m}.$$

Thus, $\lim_{N\to\infty} \mathbf{P}_{\mathbb{H},N}(\mathcal{H}(\omega^1,\ldots,\omega^j)) = \mathbf{P}_{\mathbb{H},\infty}(\mathcal{H}_\infty(\omega^1,\ldots,\omega^j))$, and this is the sense in which we say that the uniform measure on N-step upper half-plane SAWs converges weakly as $N \to \infty$ to the infinite upper half-plane SAW.

There is also another equivalent way to define the infinite upper half-plane SAW through a limiting process. If we weight each $\omega \in \mathcal{H}$ by $\beta^{-|\omega|}$, then the total weight of all such walks is infinite (see Proposition 2.1.20). However, if we weight each such walk by $x^{-|\omega|}$, $x > \beta$, then the total weight becomes finite. The limit as $x \to \beta+$ of probability measures on $\mathcal{H}$ defined in such a way has been shown to exist and give the same measure as $\mathbf{P}_{\mathbb{H},\infty}$ (see [DGKLP2011]).

2.2 Conformal invariance and SLE

Most of the results in this paper are motivated by the conjecture that the infinite upper half-plane SAW has a *scaling limit* which is *conformally invariant.* In Section 2.2.1, we briefly state some important results from complex analysis which we will be using. In section 2.2.2, we will state precisely the conjectures that the SAW converges to a scaling limit as the lattice spacing approaches zero and that this scaling limit be conformally invariant. We will then briefly describe Schramm's (chordal) SLE_κ along with a couple of very important properties it possesses.

2.2.1 Complex analysis

Since most of the probability measure we consider in this paper will be measures on random curves or subsets of the complex plane, it will be useful to describe some of the machinery from complex analysis which we will be using, including some results concerning complex Brownian motion. Important theorems here will be stated without proof, though the proofs of these theorems can be found in many complex analysis books, including [Lawler2008].

If D is a domain in $\mathbb{C}$, we will generally refer to a function $f : D \to \mathbb{C}$ as *analytic* or *holomorphic* if the complex derivative

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists at every $z \in D$. A *curve* in $\mathbb{C}$ will refer to a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $[a, b]$ is a closed interval which can either be of finite length or infinite length. The curve will be C^k or *smooth* if γ is C^k or infinitely differentiable.

If D, D' are domains and $f : D \to D'$ is holomorphic on D, one-to-one and onto D', then f is called a *conformal transformation*, or *conformal mapping* from D to D'.

A *standard (one-dimensional) Brownian motion* B_t with respect to the filtration $\{\mathcal{F}_t\}$ on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is a stochastic process satisfying:

(i) For each $0 < s < t$, the random variable $B_t - B_s$ is $\mathcal{F}_t$-measurable, independent of $\mathcal{F}_s$, and has a normal distribution with mean 0 and variance $t - s$.

(ii) W.p.1 (with probability one), the mapping $t \mapsto B_t$ is a continuous function.

Brownian motion satisfies the following type of *scaling*, sometimes referred to as *Brownian scaling*: If B_t is a standard one-dimensional Brownain motion, $r > 0$, then $Y_t = r^{-1/2} B_{rt}$ is also a standard one-dimensional Brownian motion.

A complex Brownian motion with respect to $\mathcal{F}_t$ is a process $B_t = B_t^1 + iB_t^2$, where B_t^1, B_t^2 are independent (one-dimensional) standard Brownian motions adapted to $\mathcal{F}_t$. Throughout this paper we will generally use $\mathbf{P}^z$ to denote the probability measure associated to B_t with $B_0 = z$. We will, however, generally write $\mathbf{P}$ for $\mathbf{P}^0$. Suppose D is a domain and $f : D \to \mathcal{C}$ is a non-constant holomorphic function.

$$\tau_D = \inf\{t \geq 0 : B_t \notin D\}.$$

Then a simple application of Itô's formula yields what is referred to as the *conformal invariance of Brownian motion*:

Theorem 2.2.1. *Suppose B_t is a complex Brownian motion starting at $z \in D$, and define*

$$S_t = \int_0^t |f'(B_r)|^2 \, dr, \qquad 0 \le t \le \tau_D.$$

Let $\sigma_s = S_s^{-1}$, i.e.

$$\int_0^{\sigma_s} |f'(B_r)|^2 \, dr = s.$$

Then

$$Y_s := f(B_{\sigma_s}), \qquad 0 \le s \le S_{\tau_D},$$

has the same distribution as that of a Brownian motion starting at $f(z)$ stopped at S_{τ_D}.

This shows that the law of Brownian motion is invariant (up to reparametrization) under conformal transformations. A more apt way of thinking about this for our purposes is as follows: Let D be a simply connected domain in $\mathbb{C}$ such that ∂D is smooth, and $f : D \to D'$ be a conformal transformation. For $z \in D$, let $\mu^{\#}(D, z, \partial D)$ be the probability measure on curves $\gamma : [0, t_\gamma] \to D$ with $\gamma(0) = z$, $\gamma(t_\gamma) \in \partial D$, where we consider two curves to be the same (equivalent) if one can be obtained from the other through a reparametrization, which is induced by complex Brownian motion started at D and stopped the first time it reaches ∂D (for example, if $A \subset D$ is a sufficiently nice set with $z \notin A$, then $\mu^{\#}(D, z, \partial D)\{\gamma[0, t_\gamma] \cap A = \emptyset\} = \mathbf{P}^z\{B[0, \tau_D] \cap A = \emptyset\}$). If $f \circ \mu^{\#}(D, z, \partial D)$ denotes the image of the measure $\mu^{\#}(D, z, \partial D)$ under f, then Theorem 2.2.1 tells us that

$$f \circ \mu^{\#}(D, z, \partial D) = \mu^{\#}(D', f(z), \partial D') \tag{2.2.2}$$

An especially useful notion will be that of *harmonic measure*, which requires the notion of *regular point*

Definition 2.2.3. Suppose D is a domain with boundary ∂D. A point $z \in \partial D$ is called a *regular point* (for D) if

$$\mathbf{P}^z\{\tilde{\tau}_D = 0\} = 1,$$

where $\tilde{\tau}_D = \inf\{t > 0 : B_t \notin D\}$.

Definition 2.2.4. If D is a domain such that ∂D has at least one regular point and $z \in D$, then *harmonic measure in D from z* is the probability measure on ∂D, $\mathrm{hm}(z, D; \cdot)$, given by

$$\mathrm{hm}(z, D; V) = \mathbf{P}^z\{B_{\tau_D} \in V\}.$$

We will say that ∂D is *locally analytic* if there exists a one-to-one analytic function $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = z$ such that

$$f(\mathbb{D}) \cap D = f(\{z \in \mathbb{D} : \mathrm{Im}(z) > 0\}).$$

We will say that ∂D is *piecewise analytic* if ∂D is locally analytic, except perhaps at a finite number of points.

Definition 2.2.5. If ∂D is piecewise analytic, then it has been shown that $\mathrm{hm}(z, D; \cdot)$ is absolutely continuous with respect to Lebesgue measure (length). We call the density of $\mathrm{hm}(z, D; \cdot)$ with respect to length the *Poisson kernel*, and denote it by $H_D(z, w)$.

Thus, if $u(z)$ is a harmonic function in the domain D with piecewise analytic boundary, and boundary values $u(z) = F(z)$ on ∂D, then a well known result is that

$$u(z) = \int_{\partial D} F(w) H_D(z, w)\ |dw|, \tag{2.2.6}$$

where $|dw|$ represents length measure.

A domain $D \subset \mathbb{C}$ is called *simply connected* if $\hat{\mathbb{C}} \setminus D$ is a connected subset of the Riemann sphere $\hat{\mathbb{C}}$. Equivalently, D is simply connected if and only if the region bounded by every simple closed curve $\gamma[a, b] \to D$ is contained in D.

An important driving force behind the entire theory of conformally invariant processes is the *Riemann mapping Theorem*:

Theorem 2.2.7 (Riemann mapping Theorem)**.** *Let D be a simply connected domain other than $\mathbb{C}$ and $w \in D$. Then there exists a unique conformal transformation $f : D \to \mathbb{D}$ with $f(w) = 0$, $f'(w) > 0$.*

A closed curve $\gamma : [a, b] \to \mathbb{C}$ is called a *Jordan curve* if it is one-to-one on $[a, b)$. A bounded domain D is called a *Jordan domain* if ∂D is a Jordan curve. Jordan domains are simply connected.

Proposition 2.2.8. *If D,D' are Jordan domains and z_1, z_2, z_3 and z_1', z_2', z_3' are points on ∂D,$\partial D'$, respectively, oriented counterclockwise, then there is a unique conformal transformation $f : D \to D'$, that can be extended to a homeomorphism from $\overline{D}$ to $\overline{D}'$ such that $f(z_1) = z_1'$, $f(z_2) = z_2'$, $f(z_3) = z_3'$.*

A *compact hull* K is a compact, connected subset of $\mathbb{C}$ larger than a single point such that $\mathbb{C} \setminus K$ is connected. For any compact hull K, there is a unique conformal map $F_K : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ such that $\lim_{z\to\infty} F_K(z)/z > 0$. For example, if $0 \in K$, we define $F_K(z) = 1/f_K(1/z)$, where f_K is the conformal transformation from $\mathbb{D}$ onto the image of $\mathbb{C} \setminus K$ under the map $1/z$, with $f_K(0) = 0$, $f'(0) > 0$. We define the *(logarithmic) capacity*, $\text{cap}(K)$, by $\text{cap}(K) = -\log\ f_K'(0) = \log[\lim_{z\to\infty} F_K(z)/z]$. Thus, the capacity of a compact hull K is defined in such a way that $F_K(z) \sim e^{\text{cap}(K)}z$ as $z \to \infty$. We will call a bounded subset $A \subset \mathbb{H}$ a *compact* $\mathbb{H}$*-hull* if $A = \mathbb{H} \cap \overline{A}$ and $\mathbb{H} \setminus A$ is simply connected. From here on, we will denote the set of compact $\mathbb{H}$-hulls by $\mathcal{A}$.

Proposition 2.2.9. *For each $A \in \mathcal{A}$, there is a unique conformal transformation $g_A : \mathbb{H} \setminus A \to \mathbb{H}$ such that*

$$\lim_{z\to\infty} [g_A(z) - z] = 0.$$

Note that for each $A \in \mathcal{A}$, g_A has the expansion

$$g_A(z) = z + b_1 z^{-1} + \cdots \qquad (2.2.10)$$

near infinity.

Definition 2.2.11. If $A \in \mathcal{A}$, the *half-plane capacity (from infinity)* hcap(A), is defined by

$$\mathrm{hcap}(A) = \lim_{z\to\infty} z[g_A(z) - z].$$

In other words, the half-plane capacity is taken to be the b_1 coefficient from the expansion (2.2.10), i.e.

$$g_A(z) = z + \frac{\mathrm{hcap}(A)}{z} + O\left(\frac{1}{|z|^2}\right), \qquad z \to \infty.$$

2.2.2 Scaling limit of SAW and SLE_κ

As is stated previously, all of the results described throughout this paper are motivated by the conjecture that SAW converges to a scaling limit which is conformally invariant in the limit as the lattice spacing goes to zero. Although the idea of a conformally invariant scaling limit might be something simple to grasp by looking at a picture, here we will describe the process in full mathematical detail, stating conjectures precisely. In the case of self-avoiding walk, this convergence to a scaling limit can be described as follows: Let D be a bounded, simply connected domain in $\mathbb{C}$, and let $z, w \in \partial D$. We will consider self-avoiding walks on the lattice $\delta\mathbb{Z}^2 = \delta\mathbb{Z} + i\delta\mathbb{Z}$ for $\delta > 0$. Let $[z], [w]$ be the lattice sites which are the closest to z, w for a given $\delta > 0$. Some convention needs to be taken if there are more than one lattice sites at equal distance from z, w. Any convention can be used.

Let $\mathcal{S}(D, z, w, \delta)$ denote the set of all SAWs of any length beginning at $[z]$ and ending at $[w]$, but otherwise staying inside of D. Let $\mu_{SAW}(D, z, w, \delta)$ denote the measure on $\mathcal{S}(D, z, w, \delta)$ obtained by setting

$$\mu_{SAW}(D, z, w, \delta)[\omega] = \beta^{-|\omega|},$$

for all $\omega \in \mathcal{S}(D,z,w,\delta)$. Note that the total weight of $\mu_{SAW}(D,z,w,\delta)$ is given by

$$|\mu_{SAW}(D,z,w,\delta)| = \sum_{\omega \in \mathcal{S}(D,z,w,\delta)} \beta^{-|\omega|},$$

which is finite for bounded, simply connected D. Thus we can talk about the probability measure

$$\mu^{\#}_{SAW}(D,z,w,\delta) = \frac{\mu_{SAW}(D,z,w,\delta)}{|\mu_{SAW}(D,z,w,\delta)|}.$$

We think of each $\omega \in \mathcal{S}(D,z,s,\delta)$ as a continuous curve in D, connecting $[z]$ to $[w]$, and of $\mu^{\#}_{SAW}(D,z,w,\delta)$ as a probability measure on all such continuous curves, which assigns measure 0 to curves not in $\mathcal{S}(D,z,w,\delta)$. The following conjecture is widely believed to be true, though a full proof remains elusive.

Conjecture 2.2.12 (Scaling limit of SAW)**.** *If D is a bounded, simply connected domain, $z, w, \in \partial D$, then there is a constant $b > 0$, referred to as the* boundary scaling exponent *for self-avoiding walks, a function $C(D,z,w)$, and a measure, $m_{SAW}(D,z,w)$, on continuous curves $\gamma : [0,t_\gamma] \to D$ such that $\gamma(0) = z$, $\gamma(t_\gamma) = w$, $\gamma(0,t_\gamma) \subset D$, for which*

$$\lim_{\delta \to 0+} \delta^{2b} \mu_{SAW}(D,z,w,\delta) = m_{SAW}(D,z,w),$$

where the convergence taking place is that of convergence in distribution. In other words, if E is an event of simple curves in D from z to w, then the above equation states that

$$\lim_{\delta \to 0+} \delta^{2b} \mu_{SAW}(D,z,w,\delta)[E] = m_{SAW}(D,z,w)[E].$$

Furthermore, we have

$$\lim_{\delta \to 0+} \delta^{2b} |\mu_{SAW}(D,z,w,\delta)| = C(D,z,w).$$

By construction, $C(D,z,w)$ is the total mass of $m_{SAW}(D,z,w)$.

It is conjectured that the value of the boundary scaling exponent b is $b = 5/8$. Note then, that Conjecture 2.2.12 can be stated in terms of the probability measure $\mu^{\#}_{SAW}(D, z, w, \delta)$. We have

$$\lim_{\delta \to 0+} \mu^{\#}_{SAW}(D, z, w, \delta) = m^{\#}_{SAW}(D, z, w), \tag{2.2.13}$$

where

$$m^{\#}_{SAW}(D, z, w) = \frac{m_{SAW}(D, z, w)}{C(D, z, w)}.$$

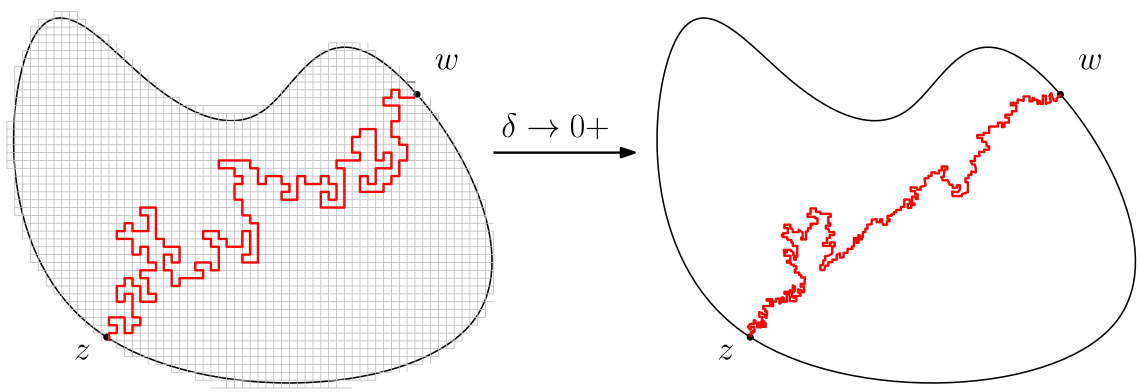

FIGURE 2.3. The self-avoiding walk in the domain D from z to w in the scaling limit as $\delta \to 0+$.

We now precisely state what we mean when we say that measure $\mu^{\#}_{SAW}$ is conformally invariant. We state the conjecture in terms of general domains D, but it has been shown in [KL2011] that the conjecture belows fails for many domains due to lattice effects that persist in the scaling limit. However, it is believed that the below conjecture is true, as stated, if we assume that the boundary of D consists of horizontal and vertical lines.

Conjecture 2.2.14 (Conformal invariance of $\mu^{\#}_{SAW}(D, z, w)$). *The measure $\mu_{SAW}(D, z, w)$ satisfies the following form of* conformal covariance*: If $f : D \to D'$ is a conformal transformation with $f(z) = z'$, $f(w) = w'$, then the image of the measure*

$m_{SAW}(D,z,w)$ *under* f*, denoted* $f \circ m_{SAW}(D,z,w)$*, satisfies*

$$f \circ m_{SAW}(D,z,w) = |f'(z)|^b |f'(w)|^b m_{SAW}(D',z',w'). \tag{2.2.15}$$

It follows then that the probability measure $m^{\#}_{SAW}$ *must be* conformally invariant*:*

$$f \circ m^{\#}_{SAW}(D,z,w) = m^{\#}_{SAW}(D',z',w').$$

Notice that we cannot use the above construction to find a scaling limit for SAW in $\mathbb{H}$ connecting 0 to ∞, or in any other unbounded domain. However, we can find the law in $\mathbb{H}$ from 0 to ∞ by first taking the limit on the uniform measures on N-step upper half-plane SAWs as $N \to \infty$, on $\delta\mathbb{Z}^2$, as in section 2.1.2, in order to obtain the measure $\mu^{\#}_{SAW}(\mathbb{H},0,\infty,\delta)$, and then by taking

$$\lim_{\delta \to 0+} \delta^{2b} \mu^{\#}_{SAW}(\mathbb{H},0,\infty,\delta) = m^{\#}_{SAW}(\mathbb{H},0,\infty).$$

The most natural question to ask then, is if SAW converges to a scaling limit which satisfies conformal invariance, is it possible to characterize this scaling limit? In [LSW2002] it was shown that if SAW converges to a scaling limit which is conformally invariant, then that scaling limit must be given by the Schramm-Loewner evolution, SLE_κ, with $\kappa = 8/3$. What we have described in this section is chordal SAW, i.e. self-avoiding walks which start at a boundary point in D and end at another boundary point in D. Thus, the scaling limit of chordal SAW is conjectured to be *chordal* $SLE_{8/3}$. Chordal SLE_κ in $\mathbb{H}$, from 0 to ∞ is defined by solving for Loewner chains with the chordal Lowener differential equation, with Brownian motion as the driving function. That is, one solves for the family of conformal maps g_t by solving the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \qquad g_0(z) = z,$$

where B_t is a standard one-dimensional Brownian motion. For $0 < \kappa \leq 4$, the paths generated by chordal SLE_κ are simple curves $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ with $\gamma(0) = 0$ and

$\gamma(0,\infty) \subset \mathbb{H}$. The conformal maps $g_t(z)$ then take $\mathbb{H}\backslash\gamma(0,t]$ onto $\mathbb{H}$ with $g_t(\gamma(t)) = 0$ and

$$g_t(z) = z + \frac{b(t)}{z} + O(\frac{1}{|z|^2}), \qquad z \to \infty,$$

where $b(t) = \text{hcap}(\gamma(0,t])$. Thus, for $0 < \kappa \le 4$, one can think of SLE_κ as the random chain of conformal maps g_t (conformal analysts) or one can equivalently think of it as the random measure on paths γ connecting 0 to ∞ in $\mathbb{H}$ (probabilists). For $\kappa > 4$, SLE_κ is still generated by paths γ, though the curves are not simple and SLE_κ is then thought of as a randomly growing set of compact $\mathbb{H}$-hulls. To be more precise, for each $z \in \overline{\mathbb{H}}$, there is a time $\tau = \tau(z) \in [0,\infty]$ such that the solution $g_t(z)$ exists for $t \in [0,\tau]$ and $\lim_{t\to\tau-} g_t(z) = \sqrt{\kappa}B_\tau$ if $\tau < \infty$. The evolving hull of the Loewner evolution is then defined to be $K_t := \{z \in \overline{H} : \tau(z) \le t\}$, $t \ge 0$. Then $K_t \in \mathcal{A}$ and one can show that g_t is the unique conformal map $\mathbb{H} \setminus K_t \to \mathbb{H}$ with $g_t(z) \sim z$ as $z \to \infty$. In the case of $\kappa \le 4$, K_t turns out to be a simple curve $\gamma[0,t]$. To say that K_t is generated by a curve for $4 < \kappa \le 8$ means that there is a curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ with $\gamma(0) \in \mathbb{R}$ such that if H_t is the unbounded component of $\mathbb{H}\backslash\gamma(0,t]$, then $K_t = \mathbb{H}\backslash H_t$.

We define SLE_κ to be conformally invariant. That is, if D is a simply connected domain (other than all of $\mathbb{C}$), and z, w are distinct points on ∂D, let $F : D \to \mathbb{H}$ be a conformal transformation with $F(z) = 0$, $F(w) = \infty$. This map is not unique, but any other such map can be written as rF for $r > 0$. We *define* chordal SLE_κ in D, from z to w to be the collection of maps $h_t(z) = F^{-1}[g_t(F(z))]$, where $g_t(z)$ is chordal SLE_κ in $\mathbb{H}$ from 0 to ∞. If $\hat{F} = rF$, for $r > 0$, then we would have $\hat{h}_t(z) = \hat{F}^{-1}[g_t(\hat{F}(z))] = F^{-1}[r^{-1}g_t(rF(z))] = F^{-1}[\hat{g}_{t/r^2}(F(z))]$, which is simply a time change of SLE_κ. Therefore, we consider SLE_κ to be a measure on curves modulo time changes.

The statement of the previous paragraph relies on the following fact about SLE_κ, which we will use at certain points throughout the paper.

Proposition 2.2.16 (SLE scaling)**.** *Suppose g_t is a chordal SLE_κ and $r > 0$. Then*

$\hat{g}_t(z) := r^{-1} g_{r^2 t}(rz)$ has the distribution of an SLE_κ. Equivalently, if γ is an SLE_κ path, and $\hat{\gamma}(t) := r^{-1}\gamma(r^2 t)$, then $\hat{\gamma}$ has the distribution of an SLE_κ path.

In Section 2.3, we will briefly discuss restriction measures, and explain why the restriction property ensures that if SAW has a conformally invariant scaling limit, then this limit must be given by $SLE_{8/3}$.

2.3 Restriction Measures

If D is a simply connected domain, and $z, w \in \partial D$ are distinct points, consider the measure $\mu^{\#}_{SAW}(D, z, w, \delta)$. If $D' \subset D$ is a subdomain of D and $z, w \in \partial D'$, then $\mu^{\#}_{SAW}(D, z, w, \delta)$ conditioned on the event that $\omega \subset D'$ is simply $\mu^{\#}_{SAW}(D', z, w, \delta)$. We refer to this as the *restriction property*. Therefore, if $\mu^{\#}_{SAW}(D, z, w, \delta)$ has a conformally invariant scaling limit, we would expect it to satisfy the restriction property.

Let us consider the restriction property for SLE_κ for $0 < \kappa \leq 4$. Then the SLE_κ curve $\gamma(t)$ is simple with $\gamma(0, \infty) \subset \mathbb{H}$. Suppose $A \in \mathcal{A}$ is bounded away from 0. Let $\Phi_A(z) = g_A(z) - g_A(0)$ be the unique conformal transformation of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with $\Phi_A(0) = 0$, and $\Phi_A(z) \sim z$ as $z \to \infty$. It can be shown [Lawler2008],[LSW2003] that $0 < \mathbf{P}\{\gamma[0,\infty) \cap A = \emptyset\} < 1$ Let $V_A := \{\gamma[0,\infty) \cap A = \emptyset\}$. On the event V_A, we can consider the path $\Phi_A \circ \gamma(t)$. We say that SLE_κ satisfies the restriction property if the distribution of $\Phi_A \circ \gamma(t)$ conditioned on the event V_A is the same as (a time change of) SLE_κ. Note that since SLE_κ is a conformally invariant process, this definition of the restriction property follows from the previous definition of the restriction property in the half-plane case. Another application of conformal invariance then shows that the restriction property for SAW is then the same as the restriction property for SLE_κ given that SLE_κ satisfies the restriction property in the half plane. Thus, a likely candidate for the scaling limit of SAW will be any SLE_κ which satisfies the restriction property. In fact, it can be shown that the only SLE_κ which satisfies the restriction property is $SLE_{8/3}$ [Lawler2008],[LSW2003].

If we are considering simple curves from 0 to ∞ modulo time changes, we can specify where the curve visits by specifying those $A \in \mathcal{A}$ bounded away from 0 with $\gamma[0,\infty) \cap A = \emptyset$. Thus, the distribution of SLE_κ for $\kappa \le 4$ is given by specifying $\mathbf{P}(V_A)$ for each $A \in \mathcal{A}$ bounded away from 0. It has been shown [LSW2003],[Lawler2008], that it suffices to consider those $A \in \mathcal{A}$ for which $\overline{A} \cap \mathbb{R} \subset (0,\infty)$. It can also be shown that it then suffices to specify these probabilities for smooth Jordan hulls. We will present the next Lemma, complete with proof, since the proof is very short and easy. It is the first step in showing that $SLE_{8/3}$ is the only SLE_κ which satisfies the restriction property. The proof follows that which is found in [Lawler2008]

Lemma 2.3.1. *Suppose $\kappa \le 4$ and there is an $\alpha > 0$ such that $\mathbf{P}(V_A) = \Phi_A'(0)^\alpha$ for all $A \in \mathcal{A}$ bounded away from 0. Then SLE_κ satisfies the restriction property.*

Proof. Suppose $\mathbf{P}(V_A) = \Phi_A'(0)^\alpha$ for all $A \in \mathcal{A}$ bounded away from 0, and let $A_1, A \in \mathcal{A}$ be bounded away from 0. Then

$$\begin{aligned}\mathbf{P}\{\Phi_A \circ \gamma[0,\infty) \cap A_1 = \emptyset | \gamma[0,\infty) \cap A = \emptyset\} &= \frac{\mathbf{P}\{\Phi_A \circ \gamma[0,\infty) \cap A_1 = \emptyset, \gamma[0,\infty) \cap A = \emptyset\}}{\mathbf{P}\{\gamma[0,\infty) \cap A = \emptyset\}} \\ &= \frac{\mathbf{P}(V_{A \cup \Phi_A^{-1}(A_1)})}{\Phi_A'(0)^\alpha}.\end{aligned}$$

But $\Phi_{A \cup \Phi_A^{-1}(A_1)} = \Phi_{A_1} \circ \Phi_A$, so the numerator is $\Phi_{A_1}'(0)^\alpha \Phi_A'(0)^\alpha$. □

The next Theorem is of special importance regarding SAW.

Theorem 2.3.2. *$SLE_{8/3}$ satisfies the restriction property. In fact, if γ is a chordal $SLE_{8/3}$ curve in $\mathbb{H}$ and $A \in \mathcal{A}$ is bounded away from 0, then*

$$\mathbf{P}\{\gamma[0,\infty) \cap A = \emptyset\} = \Phi_A'(0)^{5/8}. \tag{2.3.3}$$

This Theorem can be proved without relying on the converse of Lemma 2.3.1, although the converse to that statement is important in the study to restriction measures.

Generally speaking, chordal restriction measures are measures defined on curves in a domain D which go from boundary point to boundary point, which satisfy a more general version of the restriction formula. That is to say, if D is a domain, $z, w \in \partial D$ are distinct, and if $m(D, z, w)$ is a measure on curves $\gamma : [0, t_\gamma] \to \overline{D}$ with $\gamma(0) = z$, $\gamma(t_\gamma) = w$ and $\gamma(0, t_\gamma) \subset D$, then if D' is a subdomain of D with $z, w \in \partial D'$, then $m(D, z, w)$ satisfies the restriction property if $m(D, z, w)$, restricted to those curves that lie in D', is $m(D', z, w)$. Of course, if we want to study probability measures which satisfy the restriction property, some normalization is involved.

Restriction measures have been studied extensively (see [LSW2003],[AC2010]), and it turns out that the only restriction measure on simple curves is given by the law of $SLE_{8/3}$. To understand the behavior of general restriction measures, let us restrict our attention to probability measures on *unbounded hulls in* $\mathbb{H}$.

$D \subset \mathbb{H}$ is a *right-domain* if it is simply connected and $\partial D \cap \mathbb{R} = [0, \infty)$. Similarly, a simply connected $D \subset \mathbb{H}$ is a *left-domain* if $\partial D \cap \mathbb{R} = (-\infty, 0]$. Let $\mathcal{J}$ denote the set of closed sets K such that $K = \overline{K \cap \mathbb{H}}$ and such that $\mathbb{H} \setminus K$ is the disjoint union of a right-domain and a left-domain. An element $K \in \mathcal{J}$ is referred to as an *unbounded hull in* $\mathbb{H}$. We will sometimes refer to these as *restriction hulls*.

For example, suppose $\gamma^1, \ldots, \gamma^n : (0, \infty) \to \mathbb{H}$ are curves with $\lim_{t \to 0+} \gamma^k(t) = 0$, $\lim_{t \to \infty} \gamma^k(t) = \infty$. Let $D = \mathbb{H} \setminus [\gamma^1(0, \infty) \cup \cdots \cup \gamma^n(0, \infty)]$. Let D^+ be the connected component of D, the boundary of which contains the positive real axis, and let D^- be the connected component of D, the boundary of which contains the negative real axis. Then D^+ is a right domain and D^- is a left domain. We call $K = \mathbb{H} \setminus (D^+ \cup D^-) \in \mathcal{J}$ the *hull generated by* $\gamma^1, \ldots, \gamma^n$.

If $A \in \mathcal{A}$ is bounded away from 0, let $V_A = \{K \in \mathcal{J} : K \cap A = \emptyset\}$. We then let $\mathcal{V} = \{V_A : A \in \mathcal{A}$ is bounded away from $0\}$. It is easy to see that $\mathcal{V}$ is a π-system, so that if two probability measures $\mathbf{P}$ and $\mathbf{P}'$ agree on $\mathcal{V}$, then $\mathbf{P} = \mathbf{P}'$ on $\sigma(\mathcal{V})$. Therefore, when we refer to a measure on $\mathcal{J}$, we will really be referring to a measure on $\sigma(\mathcal{V})$.

As before, if $A \in \mathcal{A}$ is bounded away from 0, let Φ_A denote the unique conformal tranformation from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ such that $\Phi_A(0) = 0$ and $\Phi_A(z) \sim z$ as $z \to \infty$. If ν is a measure supported on V_A, let $\Phi_A \circ \nu$ denote the measure

$$\Phi_A \circ \nu(V_{A'}) = \nu\{K : \Phi_A(K \cap \mathbb{H}) \cap A' = \emptyset\}.$$

A measure ν on $\mathcal{J}$ will be called *scale-invariant* if $\nu(V_{rA}) = \nu(V_A)$ for all $A \in \mathcal{A}$ bounded away from 0. A probability measure $\mathbb{P}$ on $\mathcal{J}$ is a *restriction measure* if it is scale-invariant and for every $A \in \mathcal{A}$ bounded away from 0,

$$\Phi_A \circ \nu_A = \mathbb{P},$$

where ν_A is the conditional probability distribution given V_A.

Proposition 2.3.4. *If $\mathbb{P}$ is a restriction measure on $\mathcal{J}$, then there exists $0 \leq \alpha < \infty$ such that for each $A \in \mathcal{A}$ bounded away from 0,*

$$\mathbb{P}(V_A) = \Phi_A'(0)^{\alpha}.$$

If such a restriction measure exists, we denote it by $\mathbb{P}_\alpha$.

Note that the above proposition shows that the measure on $SLE_{8/3}$ curves is a restriction measure. In fact, it is $\mathbb{P}_{5/8}$. The proof of other restriction measures is complicated. One must construct the restriction measures carefully. It is done by considering an SLE_κ curve for $0 < \kappa \leq 8/3$ and an independent realization of the Brownian bubble soup (see, e.g [Lawler2008]) with intensity parameter λ, where

$$\kappa = \frac{6}{2\alpha + 1}, \qquad \lambda = (8 - 3\kappa)\alpha.$$

The Brownian bubble soup is a Poisson point process of closed loops which lie in $\mathbb{H}$. The restriction measure $\mathbb{P}_\alpha$ is realized by considering the hull generated by the union of the SLE_κ curve with those (filled in) loops which intersect it. It should be clear that these hulls are unbounded $\mathbb{H}$-hulls, or restriction hulls, and that $\mathbb{P}_\alpha$ is a measure on such hulls. In general, if we let (D, z, w) denote the triple consisting of a Jordan domain D, and $z, w \in \partial D$, then we can define (chordal) restriction measures on (D, z, w) through conformal transformation, much like we did with SLE_κ. We will denote restriction (probability) measures on (D, z, w) by $\mathbb{P}_\alpha^{(D,z,w)}$. Such restriction measures are then essentially characterized by the following two properties:

- **Restriction property**. If $D \subset D'$ and ∂D,$\partial D'$ agree in neighborhoods of z, w, then $\mathbb{P}_\alpha^{(D,z,w)}$ is $\mathbb{P}_\alpha^{(D',z,w)}$, conditioned on the event that the hulls K lie in D.

- **Conformal invariance**. If $f : D \to D'$ is a conformal transformation, then

$$f \circ \mathbb{P}_\alpha^{(D,z,w)} = \mathbb{P}_\alpha^{(D',f(z),f(w))}.$$

It is perhaps worth noting that one can uniquely define restriction measures which are not probability measures and carry a unique total mass through conformal covariance. Then the restriction property becomes one of actual restriction to a subdomain, as opposed to one of a conditional probability. The conformal invariance property then becomes a conformal covariance property. However, we will deal solely with the probability measures in this paper.

Chapter 3
Conditioning restriction measures on bridge heights

In Section 2.1.2 we developed an object called the infinite upper-half plane SAW, a measure on infinite length self-avoiding walks staying in the upper half-plane for all time > 0. We will say that $\omega \in \mathcal{H}$ has a *bridge point*, or *cut point*, at height $y \in \mathbb{N}$ if ω can be written $\omega = \tilde{\omega} \oplus \hat{\omega}$, where $\tilde{\omega} \in \mathcal{B}$, $\hat{\omega} \in \mathcal{H}$, and $h(\tilde{\omega}) = y$, where for a given $\omega \in \mathcal{B}$, $h(\omega)$ denotes the *height* of ω, which in the case of a bridge coincides with the span of ω. Geometrically speaking, this means that $\omega \in \mathcal{H}$ has a bridge point at height y if the horizontal line at height $y - 1/2$ intersects ω only once, where here we are thinking of ω as a sequence of sites along with the nearest neighbor bonds connecting those sites.

Let $S_y := \{z \in \mathbb{H} : 0 < \mathrm{Im}(z) < y\}$ denote the strip of height y, for $y \in \delta\mathbb{N}$. Let $\partial S_y^+ := \{z \in \mathbb{H} : \mathrm{Im}(z) = y\}$ denote the upper boundary of the strip. For $x \in \delta\mathbb{Z}$, let $\mu_{SAW}^{\#}(S, 0, x + iy, \delta)$ be defined as in section 2.2.2, and define

$$\mu_{SAW}^{\#}(S_y, 0, \partial S_y^+, \delta) = \frac{\sum_{x \in \delta\mathbb{Z}} \mu_{SAW}(S_y, 0, x + iy, \delta)}{\sum_{x \in \delta\mathbb{Z}} |\mu_{SAW}(S_y, 0, x + iy, \delta)|},$$

i.e. the probability measure on self-avoiding walks in the strip S_y, beginning at 0 and ending anywhere on the upper boundary. The following Theorem was proven in [DGKLP2011]:

Theorem 3.0.1. *Let y be a positive integer. If we condition $\mathbf{P}_{\mathbb{H},\infty}$ on the event that the walk has a bridge point at height $y - 1$ and only consider the walk up to height y, then the resulting probability measure is $\mu_{SAW}^{\#}(S_y, 0, \partial S_y^+, 1)$.*

The main purpose of this chapter is to show that this result still holds in the (conjectured) scaling limit. We will show in section 3.1 that the same result holds for restriction measures of all restriction parameters $\alpha \in [5/8, 1)$. In section 3.2 we will show that a more general version of the Theorem holds for restriction measures with $\alpha \in [5/8, 1)$.

3.1 Conditioning on the event that a restriction measure has a bridge point at a given point

In this section we will show that if one takes a restriction measure with restriction parameter α ($5/8 \le \alpha < 1$), conditions on the event that there is a bridge point at $z = x+iy$, $x, y \in \mathbb{R}$, and only considers the hull up to height y, then one obtains the law for the restriction hull on the triple $(S_y, 0, x+iy)$, where $S_y = \{z \in \mathbb{H} : 0 < \mathrm{Im}\,(z) < y\}$.

The results of this section depend heavily on the results obtained in [AC2010], where it is shown that for restriction measures $\mathbb{P}_\alpha$, with $\alpha \in [5/8, 1)$, bridge points exist $\mathbb{P}_\alpha$-a.s. Here we define a bridge point for a restriction hull K on the triple $(\mathbb{H}, 0, \infty)$ to be a point $z \in \mathbb{H}$ such that the horizontal line $y = \mathrm{Im}(z)$ intersects K in the singleton $\{z\}$. Throughout this section, we will let $C = C(K)$ denote the (random) set of bridge heights for a restriction hull K.

Given a simply connected domain $D \subset \mathbb{C}$ (not the entire complex plane), and $z, w \in \partial D$, as before, we let $\mathbb{P}_\alpha^{(D,z,w)}$ denote the law for the restriction hull with parameter $\alpha \in [5/8, 1)$, on the triple (D, z, w). Recall that the measures $\mathbb{P}_\alpha^{(D,z,w)}$ are characterized by the following two properties:

- **Conformal Invariance:** If $f : D \to D'$ is a conformal transformation from D onto a simply connected domain D', then if $f \circ \mathbb{P}_\alpha^{(D,z,w)}$ denotes the image of $\mathbb{P}_\alpha^{(D,z,w)}$ under f then
$$f \circ \mathbb{P}_\alpha^{(D,z,w)} = \mathbb{P}_\alpha^{(D',f(z),f(w))}.$$
- **Restriction:** The conditional law of K on the triple (D, z, w), restricted to those hulls K that lie in a subdomain $D' \subset D$ with $z, w \in \partial D'$, is distributed according to the law $\mathbb{P}_\alpha^{(D',z,w)}$.

Given a triple (D, z, w), if A is such that $A = \overline{A \cap D}$, $D \backslash A$ is simply connected and $z, w \notin A$, then the probability measure $\mathbb{P}_\alpha^{(D,z,w)}$ is characterized by the values

$$\mathbb{P}_\alpha^{(D,z,w)}\{K \cap A = \emptyset\}.$$

On the canonical triple $(\mathbb{H}, 0, \infty)$, we can calculate the above probabilities using the *restriction formula:*

$$\mathbb{P}_\alpha\{K \cap A = \emptyset\} = \Phi_A'(0)^\alpha,$$

where, as in section 2.3, Φ_A is the unique conformal transformation from $\mathbb{H}\backslash A$ onto $\mathbb{H}$ such that $\Phi_A(0) = 0$ and $\Phi_A(z) \sim z$ as $z \to \infty$. Therefore, we see that on a general triple (D, z, w), the restriction formula becomes

$$\mathbb{P}_\alpha^{(D,z,w)}\{K \cap A = \emptyset\} = \Phi_{f(A)}'(0)^\alpha, \tag{3.1.1}$$

where f is a conformal transformation from D onto $\mathbb{H}$ satisfying $f(z) = 0$ and $f(w) = \infty$. We would like to define the conditional probability on the event that a restriction hull has a bridge point at a given $z \in \mathbb{H}$. Since this is an event of measure zero, we define the conditioning as follows:

$$\mathbb{P}_\alpha\{K \cap A = \emptyset | z \in C\} = \lim_{\epsilon \to 0+} \mathbb{P}_\alpha\{K \cap A = \emptyset | K \cap I(z, \epsilon) = \emptyset\}, \tag{3.1.2}$$

where $I(z, \epsilon) = \{w \in \mathbb{H} : \mathrm{Im}(w) = 1 : |w - z| \geq \epsilon\}$, i.e. the line $y = \mathrm{Im}(z)$ with a gap of width 2ϵ centered at z removed. Let us now state the main theorem of this section:

Theorem 3.1.3. *Let K be a restriction hull under the law $\mathbb{P}_\alpha := \mathbb{P}_\alpha^{(\mathbb{H},0,\infty)}$. Then conditioning on the event that K has a bridge point at $z = x + iy$ and considering K up to height y gives the law of a restriction hull $\hat{K}$ on the triple $(S_y, 0, x + iy)$.*

Proof. It suffices to prove the result in the case that $z = x + i$, and then the general result will follow from scaling. Let $S := S_1 = \{z \in \mathbb{H} : 0 < \mathrm{Im}(z) < 1\}$. Let $I(x + i, \epsilon) = \{z \in \mathbb{H} : \mathrm{Im}(z) = 1, |z - (x + i)| \geq \epsilon\}$ be the horizontal line $y = 1$

with a gap of width 2ϵ centered at $x+i$ removed. Let A be such that $A=\overline{S\cap A}$, $S\backslash A$ is simply connected and A is bounded away from $0, x+i$. We want to show that

$$\mathbb{P}_\alpha\{K\cap A=\emptyset|x+i\in C\}=\mathbb{P}_\alpha^{(S,0,x+i)}\{K\cap A=\emptyset\}.$$

Since the event that $x+i$ is in C is an event of measure 0, we must define the conditional probability in terms of a limit:

$$\mathbb{P}_\alpha\{K\cap A=\emptyset|x+i\in C\}=\lim_{\epsilon\to 0+}\mathbb{P}_\alpha\{K\cap A=\emptyset|K\cap I\left(x+i,\epsilon\right)=\emptyset\}.$$

Let $E_\epsilon=A\cup I\left(x+i,\epsilon\right)$. Then the conditional probability can be written as

$$\lim_{\epsilon\to 0+}\mathbb{P}_\alpha\{K\cap A=\emptyset|K\cap I\left(x+i,\epsilon\right)=\emptyset\}=\lim_{\epsilon\to 0+}\frac{\mathbb{P}_\alpha\{K\cap E_\epsilon=\emptyset\}}{\mathbb{P}_\alpha\{K\cap I\left(x+i,\epsilon\right)=\emptyset\}}.$$

In [AC2010], it is shown that

$$\mathbb{P}_\alpha\{K\cap I\left(x+i,\epsilon\right\}=\emptyset\}\sim\frac{\pi^{2\alpha}}{16^\alpha\cosh^{2\alpha}\left(\pi x/2\right)}\epsilon^{2\alpha}.$$

as $\epsilon\to 0+$. Thus, by the restriction formula (3.1.1), the problem reduces to showing that

$$\mathbb{P}_\alpha\{K\cap E_\epsilon=\emptyset\}\sim\frac{\pi^{2\alpha}}{16^\alpha\cosh^{2\alpha}\left(\pi x/2\right)}\epsilon^{2\alpha}\Phi'_{f(A)}\left(0\right)^\alpha, \tag{3.1.4}$$

as $\epsilon\to 0+$, where f is the conformal transformation from S onto $\mathbb{H}$ such that $f\left(0\right)=0$, $f\left(x+i\right)=\infty$. Given $x'\in\left(x-\epsilon,x+\epsilon\right)$, we write $f_{x'}$ for the conformal transformation from S onto $\mathbb{H}$ with $f_{x'}\left(0\right)=0$ and $f_{x'}\left(x'+i\right)=\infty$. It is clear that

$$\Phi'_{f_{x'}(A)}\left(0\right)\sim\Phi'_{f(A)}\left(0\right)$$

as $\epsilon\to 0+$. We will use the Brownian excursion method described in section 2.3 to calculate $\mathbb{P}_\alpha\{K\cap E_\epsilon=\emptyset\}$. That is, if $\hat{B}_t$ is a complex Brownian motion conditioned to stay in $\mathbb{H}$ for all $t>0$, then we have [Virág2003]

$$\mathbf{P}\{\hat{B}[0,\infty)\cap A=\emptyset\}=\Phi'_A(0). \tag{3.1.5}$$

To that end, let $\hat{B}_t = X_t + i\hat{Y}_t$, $t \geq 0$, be a Brownian excursion in $\mathbb{H}$, starting at 0 and going to ∞. Then X_t is a standard one dimensional Brownian motion and $\hat{Y}_t$ is a Bessel-3 process. In [Virág2003], it is shown that the probability that the Brownian excursion misses E_ϵ is given by $\Phi'_{E_\epsilon}(0)$. Thus, we must show that

$$\Phi'_{E_\epsilon}(0) \sim \frac{\pi^2}{16\cosh^2(\pi x/2)}\epsilon^2 \Phi'_{f(A)}(0).$$

Given $r > 0$, let $\hat{\sigma}_r = \inf\{t \geq 0 : \hat{Y}_t = r\}$ be the first time that the Brownian excursion $\hat{B}_t$ reaches height r.

In order for the Brownian excursion started at 0 to make it to ∞ without hitting A or $I(x+i, \epsilon)$, it must first make it to the line $\mathrm{Im}(z) = 1$ while avoiding A, it must hit the horizontal line $y = 1$ somewhere in the gap of width 2ϵ centered i, and then it must make it to ∞ while avoiding $I(x+i, \epsilon)$ and A. Starting at $x + \lambda\epsilon + i$, for some $\lambda \in [-1, 1]$, in order to make it to ∞, it first must make it up to the line $\mathrm{Im}(z) = 2$ while avoiding A and $I(x+i, \epsilon)$, and then must pass to infinity while still avoiding the two sets. Upon integration over the starting points, these three events are independent by the Strong Markov property, and therefore the probability that $\hat{B}_t$ misses E_ϵ is given by the product of the three events, integrated over the starting points. The first event is the event that $\hat{B}(0, \hat{\sigma}_1) \cap A = \emptyset$ and $X_{\hat{\sigma}_1} \in (x - \epsilon, x + \epsilon)$. This is simply the probability that a Brownian excursion in the unit strip, starting at 0 and ending somewhere along the upper boundary of the strip avoids A and exits somewhere along the gap. The exit density for B.E. on the line $\mathrm{Im}(z) = 1$ is given by $\pi/4 \cdot \cosh^{-2}(\pi x/2)$. Thus, the probability of the first event is

$$\int_{-\epsilon}^{\epsilon} \frac{\pi}{4\cosh^2(\pi(x - x')/2)} \Phi'_{f_{x'}(A)}(0)\, dx' \sim \frac{\pi}{2\cosh^2(\pi x/2)}\epsilon \Phi'_{f(A)}(0).$$

Now, the second event is the event that $\hat{B}(\hat{\sigma}_1, \hat{\sigma}_2)$ avoids A and $I(x+i)$ simultaneously, and hits the line $\mathrm{Im}(z) = 2$ anywhere along the line. Since $\hat{B}_{\hat{\sigma}_1}$ lies somewhere along the gap, we are looking at the event that the Brownian excursion, started at $x + \lambda\epsilon + i$, $\lambda \in [-1, 1]$, hits the line of height 2 while avoiding $I(x+i, \epsilon)$ and A.

We argue that this is asymptotically the same as the event that $\hat{B}(\hat{\sigma}_1, \hat{\sigma}_2)$ avoids $I(x+i,\epsilon)$. Indeed, in order for this path to hit A while avoiding $I(x+i,\epsilon)$, it must go back underneath the gap and hit A, which is an event of order ϵ, and then it must pass back through the gap and proceed to the line of height 2, which is an event of order at most ϵ. Thus, the probability of such an event is at most $O(\epsilon^2)$, and therefore doesn't contribute asymptotically (The statement that passing back through the gap and going to the line of height 2 is at most order ϵ is due to the argument that follows).

Now, the probability that the path, started at $x+\lambda\epsilon+i$, hits the line $\mathrm{Im}(z)=2$ while avoiding $I(x+i,\epsilon)$, can be calculated asymptotically. The exit density along the line $y=2$ of a Brownian motion starting at $x+\lambda\epsilon+i$ in the region $S_{\epsilon,x} := (\mathbb{R}\times[0,2i])\setminus I(x+i,\epsilon)$ is asymptotically the same as the (translated) exit density along the line $y=2i$ of a Brownian motion started at $\lambda\epsilon+i$ in the region $S_\epsilon := (\mathbb{R}\times[0,2i])\setminus I(i,\epsilon)$ up to translation which, upon integration over the line $y=2$, is immaterial. Asymptotically, this is the same as the exit density of a Brownian motion along the line $y=2$, started at $\lambda\epsilon+i$, conditioned to avoid the real axis, in the same region. In other words, $H_{S_\epsilon}(z', x+iy)$ is the Radon-Nikodym derivative of the probability measure $\mu^{\#}_{S_\epsilon}(z', \partial S_\epsilon)$, which is the probability measure on curves starting at z' and ending anywhere on ∂S_ϵ, derived from Brownian motion started at z' and stopped at τ_{S_ϵ}. Then, asymptotically, the event that a Brownian motion started at $z'=\lambda\epsilon+i$ goes back down through the gap of width 2ϵ a distance of order 1 is $O(\epsilon)$, and the probability that it then comes back up through the gap to exit S_ϵ in a region of $x+iy$ is at least $O(\epsilon)$, for a combined probability of at least $O(\epsilon^2)$, which doesn't contribute asymptotically. The above exit density, $H_{S_\epsilon}(z', x+iy)$, is given by [AC2010]

$$H_{S_\epsilon}(\lambda\epsilon+i, y+2i) \sim \frac{\pi\sqrt{1-\lambda^2}}{8\cosh^2(\pi y/2)}\epsilon.$$

Therefore, the probability that $\hat{B}(\hat{\sigma}_1, \hat{\sigma}_2)$ avoids $I(x+i,\epsilon)$, is asymptotically given by

$$\int_{-1}^{1}\int_{-\infty}^{\infty}\frac{\pi\sqrt{1-\lambda^2}}{8\cosh^2(\pi y/2)}\epsilon dy d\lambda = \frac{\pi}{8}\frac{\pi}{2}\frac{4}{\pi}\epsilon$$
$$= \frac{\pi}{4}\epsilon.$$

Putting together what we have so far, we see that the event that the Brownian excursion reaches the line $y = 2i$ while avoiding E_ϵ is asymptotically given by

$$\frac{\pi^2}{8\cosh^2(\pi x/2)}\epsilon^2\phi'_{f(A)}(0).$$

Now we must calculate the probability that $\hat{B}(\hat{\sigma}_2, \infty)$ passes to infinity while avoiding A and $I(x+i, \epsilon)$. But the argument given above also shows that asymptotically, this is the same as the probability that $\hat{B}(\hat{\sigma}_2, \infty)$ avoids $I(x+i, \epsilon)$. But it is clear that this is asymptotically the same as the event that $\hat{B}(\hat{\sigma}_2, \infty)$ passes to ∞ before returning to the line $y = i$. However, this is simply the probability that a Bessel-3 process started at 2 passes to ∞ before hitting 1. Recall that for a Bessel-d process, given $0 < x_1 < x < x_2$, the probability that the process hits x_2 before x_1 is given by

$$\phi_0(x; x_1 x_2) = \frac{x^{1-2a} - x_1^{1-2a}}{x_2^{1-2a} - x_1^{1-2a}}.$$

Here a and d are related by $a = (d-1)/2$. Plugging in the appropriate values, and taking the limit $x_2 \to \infty$, we see that the probability that the excursion goes to ∞ before hitting the line $y = i$ is exactly $1/2$. Multiplying with the above probability, we obtain the desired result. $\square$

3.2 Conditioning on generalized bridge points

In this section, we will expand the result from section 3.1 to show that we can condition on the event that a restriction hull has a more generalized type of "bridge point", to obtain restriction measures in domains other than the strip. We begin by considering smooth curves $\gamma : [a, b] \to \mathbb{H}$, where we allow the possibility that

$a, b = \pm\infty$, which disconnect $\mathbb{H}$ into two simply connected domains, H_1 and H_2, where $0 \in \overline{H}_1$. For example, given $r > 0$, one can take $\gamma(t) = re^{i\pi t}$, $0 \leq t \leq 1$. Then $H_1 = r\mathbb{D}_+ := \mathbb{H} \cap r\mathbb{D}$ and $H_2 = \mathbb{H} \setminus r\overline{\mathbb{D}}$.

We then consider the event that for a given $z \in \mathbb{H}$, $K \cap \gamma[a, b] = \{z\}$ for restriction hulls K under the law $\mathbb{P}_\alpha$. This should be an event of $\mathbb{P}_\alpha$-measure 0, but we argue that by the appropriate limiting process, the $\mathbb{P}_\alpha$ conditional law on this event is well defined. In what follows, we will let C_γ denote the (random) set of *generalized bridge points*; that is, $C_\gamma = \{z \in \mathbb{H} : K \cap r\gamma[a, b] = \{z\}, r > 0\}$. Note that in this case, a point $z \in \mathbb{H}$ is a generalized bridge point for the restriction hull K if K intersects some dilation of $\gamma[a, b]$ only at $\{z\}$.

Let $\mathcal{X}$ denote the set of all smooth curves $\gamma : [a, b] \to \mathbb{H}$ which disconnect the half-plane into disjoint, simply connected domains H_1, H_2, with $0 \in \overline{H}_1$. It will be useful to parametrize such curves by arc length. To that end. let $\mathcal{X}^*$ denote the set of all $\gamma \in \mathcal{X}$ which are parametrized by arc length. Given $\gamma \in \mathcal{X}^*$, $z \in \gamma[a, b]$, let $s_z \in [a, b]$ be such that $\gamma(s_z) = z$. For $\epsilon > 0$, we define the sets

$$I_\gamma(z, \epsilon) := \gamma[a, s_z - \epsilon] \cup \gamma[s_z + \epsilon, b],$$

the curve $\gamma[a, b]$ with a gap of arc length 2ϵ centered at z removed. We would like to define the conditional probability, $\mathbb{P}_\alpha\{\cdot | z \in C_\gamma\}$. Since for a given $z \in \mathbb{H}$, the event $z \in C_\gamma$ has $\mathbb{P}_\alpha$-probability zero, we define this conditioning as follows:

$$\mathbb{P}_\alpha\{\cdot | z \in C_\gamma\} = \lim_{\epsilon \to 0+} \mathbb{P}_\alpha\{\cdot | K \cap I_\gamma(z, \epsilon) = \emptyset\}.$$

Let us now state the main Theorem we would like to prove in this section.

Theorem 3.2.1. *Let K be a restriction hull under the law $\mathbb{P}_\alpha$ and let $\gamma \in \mathcal{X}^*$ be such that* $\mathrm{Im}(\gamma(z))$ *is bounded away from infinity, uniformly in z. Let $\tau = \inf\{t \geq 0 : K_t \in r\gamma[a, b]\}$. Conditioning on the event $\{z \in C_\gamma\}$ and considering K up to time τ gives the law of a restriction hull $\hat{H}$ on the triple $(H_1, 0, z)$.*

Note that Theorem 3.2.1 is a generalization of Theorem 3.1.3, since taking $\gamma(s) = s + iy$ for fixed $y > 0$, and applying Theorem 3.2.1 gives Theorem 3.1.3.

Given a simply connected domain D, $z, w \in \partial D$, let $H_{\partial D}(z, w)$ denote the boundary Poisson kernel, defined by

$$H_{\partial D}(z, w) = \lim_{\epsilon \to 0+} \epsilon^{-1} H_D(z + \epsilon \mathbf{n}_z, w), \tag{3.2.2}$$

where $\mathbf{n}_z$ denotes the inward unit normal at z. If $z \in \partial D$, $v \in D$, we let $H_D(z, v)$ denote the Poisson kernel. The distinction is given by the subscript and by whether z, w, v are boundary points or interior points. Given $\gamma \in \mathcal{X}^*$ such that $\max_{z \in \gamma[a,b]}\{\mathrm{Im}(\gamma(z))\} \leq M$ for some $M > 0$, let

$$S_{\gamma,z,\epsilon} := \{w \in \mathbb{H} : 0 < \mathrm{Im}(w) < M\} \setminus I_\gamma(z, \epsilon).$$

Before proving Theorem 3.2.1, we prove the following Proposition.

Proposition 3.2.3. *Let K be a restriction hull under the law $\mathbb{P}_\alpha$. Suppose that $\gamma \in \mathcal{X}^*$ is such that there exists $M > 0$ such that*

$$\max_{w \in \gamma[a,b]} \gamma(w) \leq M.$$

Then there exists a constant $c > 0$ such that

$$\mathbb{P}_\alpha\{K \cap I_\gamma(z, \epsilon) = \emptyset\} \sim c H_{\partial H_1}(0, z)^\alpha g(z)^\alpha \epsilon^{2\alpha}, \tag{3.2.4}$$

as $\epsilon \to 0+$, where $g(z) = \lim_{M \to \infty} \int_{-\infty}^{\infty} H_{S_{\gamma,z,\epsilon}}(z, x + iM)\, \mathrm{d}x$.

Proof. Let $\hat{B}_t$ be an $\mathbb{H}$-excursion. That is, $\hat{B}_t$ is a complex Brownian motion conditioned to stay in the half-plane. Recall that

$$\Phi'_{I_\gamma(z,\epsilon)}(0) = \mathbf{P}\{\hat{B}[0, \infty) \cap I_\gamma(z, \epsilon) = \emptyset\}.$$

Let $\gamma \in \mathcal{X}^*$ and let $\tau_\gamma = \inf\{t \geq 0 : \hat{B}_t \in \gamma[a, b]\}$. In order for $\hat{B}$ to reach ∞ while avoiding $I_\gamma(z, \epsilon)$, we must have $B_\tau \in \gamma(s_z - \epsilon, s_z + \epsilon)$; $\hat{B}$ must first hit the curve γ

somewhere along the gap of length 2ϵ, and then, starting somewhere in the gap of width 2ϵ, the $\mathbb{H}$-excursion must proceed to infinity while still avoiding $I_\gamma(z,\epsilon)$. By the strong Markov property, the probability that $\hat{B}[0,\infty)$ avoids $I_\gamma(z,\epsilon)$ is the product of the probability of these two events.

Suppose D is a simply connected domain in $\mathbb{C}$ other than all of $\mathbb{C}$, and let $z \in D$ and $\Gamma \subset \partial D$ be a smooth boundary arc. If B_t is a complex Brownian motion and $\tau_D = \inf\{t \geq 0 : B_t \in \partial D\}$, let $\mu_D(z, \partial D)$ be the measure on simple curves which start at z, end anywhere along the boundary of D, derived from B_t starting at z and stopped at τ_D. Let $\mu_D(z,\Gamma)$ be $\mu_D(z,\partial D)$ restricted to curves that end somewhere along Γ. If $w \in \partial D$ is in a smooth neighborhood of ∂D, we can define $\mu_D(z,w)$ through a limiting process by considering smooth boundary arcs Γ_ϵ of length ϵ, normalizing the measures by ϵ^{-1}. Also, if $z \in \partial D$ is in a smooth neighborhood of ∂D, we can define $\mu_{\partial D}(z,\Gamma)$ by $\mu_{\partial D}(z,\Gamma) = \lim_{\epsilon \to 0+} \epsilon^{-1} \mu_D(z + \epsilon \mathbf{n}_z, \Gamma)$, where $\mathbf{n}_z$ is the inward pointing unit normal to ∂D at z. By applying the limiting process twice, we can define a measure on simple curves from z to w, but otherwise staying in D, which is derived from Brownian motion. We then define $\mu^{\#}_{\partial D}(z,w) = \mu_{\partial D}(z,w)/|\mu_{\partial D}(z,w)|$. In fact, the measure $\mu^{\#}_{\partial D}(z,w)$ is known as Brownian excursion measure in D from z to w.

Let us first consider the probability of the first event, $\mathbf{P}\{\hat{B}_{\tau_\gamma} \in \gamma(s_z - \epsilon, s_z + \epsilon)\}$. We argue that this probability is asymptotically given by

$$\mathbf{P}\{\hat{B}_{\tau_\gamma} \in \gamma(s_z - \epsilon, s_z + \epsilon)\} \sim c_1 H_{\partial H_1}(0,z)\epsilon, \tag{3.2.5}$$

as $\epsilon \to 0+$. Consider the measure $\mu_{\partial H_1}(0, H_1^+)$ as given above, where H_1^+ is the upper boundary of H_1. That is, $H_1^+ = \gamma[a,b]$. Then we have

$$\mu_{\partial H_1}(0, \gamma(s_z - \epsilon, s_z + \epsilon)) = \int_{\gamma(s_z-\epsilon, s_z+\epsilon)} \mu^{\#}_{\partial H_1}(0,z') H_{\partial H_1}(0,z')|dz'|,$$

where we use the superscript $\#$ notation to denote that we have normalized the measure into a probability measure. It follows that the probability that a Brownian excursion in H_1 from 0 to a point on the curve $\gamma[a,b]$ lands somewhere in the gap

$\gamma(s_z - \epsilon, s_z + \epsilon)$ is given by

$$\mu^{\#}_{\partial H_1}(0, \partial H_1^+)\{\hat{\gamma}(t_{\hat{\gamma}}) \in \gamma(s_z - \epsilon, s_z + \epsilon)\} = c_1 \int_{s_z-\epsilon}^{s_z+\epsilon} H_{\partial H_1}(0, \gamma(s))\gamma'(s)ds$$
$$\sim c_1 H_{\partial H_1}(0, z)\epsilon,$$

where $c_1 = 2[\int_{\gamma[a,b]} H_{\partial D}(0, z')\ |dz'|]^{-1}$, and we are using $\hat{\gamma}$ to represent a curve from the support of $\mu^{\#}_{\partial H_1}(0, \partial H_1^+)$.

Now we must show that this probability is the same as $\mathbf{P}\{\hat{B}_{\tau_\gamma} \in \gamma(s_z - \epsilon, s_z + \epsilon)\}$. This follows from the conditioning used to obtain the measure $\mu^{\#}_{\partial H_1}(0, \partial H_1^+)$. We start a complex Brownian motion at 0 and condition that it stays above the real axis for all $t > 0$ through a limiting process, stopping it at the time it hits $\partial H_1^+ = \gamma[a, b]$. But this is the same as the measure given by $\mathbf{P}\{\hat{B}_{\tau_\gamma} \in E\}$, where E is an event of continuous curves in H_1, starting at 0 and ending somewhere along the curve $\gamma[a, b]$.

We now consider the probability $\mathbf{P}\{\hat{B}[\tau_\gamma, \infty) \cap I_\gamma(z, \epsilon) = \emptyset\}$, the probability that the $\mathbb{H}$-excursion, started after the first time it hits the gap of length 2ϵ, passes to infinity while avoiding $I_\gamma(z, \epsilon)$. We argue that this probability is asymptotically given by

$$\mathbf{P}\{\hat{B}[\tau_\gamma, \infty) \cap I_\gamma(z, \epsilon) = \emptyset\} \sim c_2 g(z)\epsilon \tag{3.2.6}$$

as $\epsilon \to 0+$, where g is as in the statement of Proposition and c_2 is a positive constant. Let $\hat{\tau}_M = \inf\{t \geq 0 : \mathrm{Im}(\hat{B}_t) = M\}$. For an $\mathbb{H}$-excursion starting at $z' \in \gamma(s_z - \epsilon, s_z + \epsilon)$, in order for the $\mathbb{H}$-excursion to pass to infinity while avoiding $I_\gamma(z, \epsilon)$, it must first reach height M while avoiding $I_\gamma(z, \epsilon)$. Then the probability that the $\mathbb{H}$-excursion started at $z' \in \gamma(s_z - \epsilon, s_z + \epsilon)$ passes to infinity while avoiding $I_\gamma(z, \epsilon)$ can be obtained by considering the probability that $\hat{B}[\tau_\gamma, \hat{\tau}_M] \cap I_\gamma(z, \epsilon) = \emptyset$ in the limit as M tends to ∞. Arguing as in the proof of Theorem 3.1.3, we have

$$\mathbf{P}\{\hat{B}[\tau_\gamma, \hat{\tau}_M] \cap I_\gamma(z, \epsilon) = \emptyset\} \sim \int_{\gamma(s_z-\epsilon, s_z+\epsilon)} \int_{-\infty}^{\infty} H_{S_{\gamma,z,\epsilon}}(z', x + iM)\ dx\ |dz'|$$
$$\sim 2\left(\int_{-\infty}^{\infty} H_{S_{\gamma,z,\epsilon}}(z, x + iM)\ dx\right)\epsilon$$

as $\epsilon \to 0+$. Thus, setting $g(z) = \lim_{M\to\infty} \int_{-\infty}^{\infty} H_{S_{\gamma,z,\epsilon}}(z, x+iM)\ dx$, we arrive at (3.2.6). Combining this result with (3.2.5), we arrive at the conclusion of the proposition. $\square$

We are now in a position to prove Theorem 3.2.1. Given a smooth curve $\gamma \in \mathcal{X}^*$ such that $\max_{w\in\gamma[a,b]} \gamma(w) \leq M$ for some $M > 0$, let $A \subset \overline{H_1}$ be such that $A = \overline{A} \cap \overline{H_1}$, $H_1 \setminus A$ is simply connected and A is bounded away from 0 and z for a fixed $z \in \gamma[a,b]$. The measure $\mathbb{P}_\alpha^{(H_1,0,z)}$ is determined by the probabilities [Lawler2008],[LSW2003]

$$\mathbb{P}_\alpha^{(H_1,0,z)}\{K \cap A = \emptyset\},$$

for such A. Using conformal invariance, the above probabilities are given by

$$\mathbb{P}_\alpha^{(H_1,0,z)}\{K \cap A = \emptyset\} = \Phi'_{f(A)}(0)^\alpha,$$

where f is a conformal transformation from H_1 onto $\mathbb{H}$ such that $f(0) = 0$ and $f(z) = \infty$. Let us now proceed to the proof of Theorem 3.2.1

Proof of Theorem 3.2.1. We prove the result in the case $r = 1$. Scaling will then give the desired result. Let $\gamma \in \mathcal{X}^*$, $M > 0$, and A be as above. Define $E_{\gamma,\epsilon} := I_\gamma(z,\epsilon) \cup A$. Note that

$$\begin{aligned}
\mathbb{P}_\alpha\{K \cap A = \emptyset | z \in C_\gamma\} &= \lim_{\epsilon\to 0+} \mathbb{P}_\alpha\{K \cap A = \emptyset | K \cap I_\gamma(z,\epsilon) = \emptyset\} \\
&= \lim_{\epsilon\to 0+} \frac{\mathbb{P}_\alpha\{K \cap A = \emptyset, K \cap I_\gamma(z,\epsilon) = \emptyset\}}{\mathbb{P}_\alpha\{K \cap I_\gamma(z,\epsilon) = \emptyset\}} \\
&= \lim_{\epsilon\to 0+} \frac{\mathbb{P}_\alpha\{K \cap E_{\gamma,\epsilon} = \emptyset\}}{\mathbb{P}_\alpha\{K \cap I_\gamma(z,\epsilon) = \emptyset\}}.
\end{aligned}$$

Therefore, it suffices to show that, as $\epsilon \to 0+$,

$$\mathbb{P}_\alpha\{K \cap E_{\gamma,\epsilon} = \emptyset\} \sim \mathbb{P}_\alpha\{K \cap I_\gamma(z,\epsilon) = \emptyset\}\mathbb{P}_\alpha^{(H_1,0,z)}\{\hat{K} \cap A = \emptyset\},$$

or, equivalently, that

$$\Phi'_{E_{\gamma,\epsilon}}(0) \sim \Phi'_{I_\gamma(z,\epsilon)}(0)\Phi'_{f(A)}(0), \tag{3.2.7}$$

where $f : H_1 \to \mathbb{H}$ is a conformal transformation with $f(0) = 0$, $f(z) = \infty$. For a given $\epsilon > 0$ and $z' \in \gamma(s_z - \epsilon, s_z + \epsilon)$, let $f_{z'} : H_1 \to \mathbb{H}$ be a conformal transformation with $f_{z'}(0) = 0$, $f_{z'}(z') = \infty$. Note that, as $\epsilon \to 0+$,

$$\Phi'_{f_{z'}(A)}(0) \sim \Phi'_{f(A)}(0).$$

We proceed to show equation (3.2.7) holds by calculating the probabilities for an $\mathbb{H}$-excursion. To that end, let $\hat{B}_t$, $t \geq 0$ be an $\mathbb{H}$-excursion. In order for $\hat{B}_t$ to pass from 0 to ∞ while avoiding $E_{\gamma,\epsilon}$, it must first pass from 0 until it hits $\gamma[a, b]$ somewhere along the gap of length 2ϵ while avoiding A, and then it must pass from a point along the gap to ∞, all while still avoiding $I_\gamma(z, \epsilon)$ and A. By the strong Markov property, these two events are independent, and therefore the probability that $\hat{B}_t$ passes from 0 to ∞ while avoiding $E_{\gamma,\epsilon}$ is the product of the probabilities of these two events. Let us first consider the probability of the first event,

$$\mathbf{P}\{\hat{B}[0, \tau_\gamma] \cap E_{\gamma,\epsilon} = \emptyset\} = \mathbf{P}\{\hat{B}[0, \tau_\gamma] \cap A = \emptyset, \hat{B}_{\tau_\gamma} \in \gamma(s_z - \epsilon, s_z + \epsilon)\}.$$

By an argument similar to that in the proof of Proposition 3.2.3, the probability of this event is given by the law $\mu^{\#}_{\partial H_1}(0, \partial H_1^+)$, and we have

$$\begin{aligned} \mathbf{P}\{\hat{B}[0, \tau_\gamma] \cap E_{\gamma,\epsilon} = \emptyset\} &= \mu^{\#}_{\partial H_1}(0, \partial H_1^+)\left(\{\hat{\gamma}[0, t_\gamma] \cap E_{\gamma,\epsilon} = \emptyset\}\right) \\ &= c_1' \int_{\gamma(s_z-\epsilon, s_z+\epsilon)} H_{\partial H_1}(0, z')\Phi'_{f_{z'}(A)}(0)\ |dz'| \\ &\sim c_1 H_{\partial H_1}(0, z)\Phi'_{f(A)}(0)\epsilon \end{aligned}$$

as $\epsilon \to 0+$, where c_1 is as in the proof of Proposition 3.2.3.

Let us now consider the probability of the second event. We argue that the probability that, starting from a point along the gap of length 2ϵ, the $\mathbb{H}$-excursion passes to ∞ while avoiding $I_\gamma(z, \epsilon)$ and A is asymptotically the same as the probability that, starting from the gap of width 2ϵ, the $\mathbb{H}$ excursion passes to ∞ while avoiding $I_\gamma(z, \epsilon)$. Indeed, by the proof of Proposition 3.2.3, there is an $O(\epsilon)$ probability that the $\mathbb{H}$-excursion passes below the gap a distance of $O(1)$ to touch A, and then another

$O(\epsilon)$ probability that it passes back up through the gap without touching $I_\gamma(z,\epsilon)$. Overall, the probability of this event is $O(\epsilon^2)$, which doesn't contribute asymptotically. Therefore, we have, as $\epsilon \to 0+$,

$$\begin{aligned}\mathbf{P}\{\hat{B}[\tau_\gamma,\infty) \cap E_{\gamma,\epsilon} = \emptyset\} &\sim \mathbf{P}\{\hat{B}[\tau_\gamma,\infty) \cap I_\gamma(z,\epsilon) = \emptyset\} \\ &\sim c_2 g(z)\epsilon,\end{aligned}$$

where c_2 and $g(z)$ are the same as in (3.2.6). Putting this together with the probability of the first event, we have

$$\begin{aligned}\mathbf{P}\{\hat{B}[0,\infty) \cap E_{\gamma,\epsilon} = \emptyset\} &\sim cH_{\partial H_1}(0,z)g(z)\Phi'_{f(A)}(0)\epsilon^2 \\ &\sim \Phi'_{I_\gamma(z,\epsilon)}(0)\Phi'_{f(A)}(0)\end{aligned}$$

as $\epsilon \to 0+$, which completes the proof of Theorem 3.2.1. □

Remark 3.2.8. It is worth noting that Theorem 3.2.1 still holds if we remove the condition that $\max_{w\in\gamma[a,b]} \mathrm{Im}(\gamma(w)) \leq M$. For example, if $\gamma \in \mathcal{X}^*$ has unbounded imaginary part, then there exists a sequence $\{\gamma_n\}$ such that $\max_{w\in\gamma_n[a_n,b_n]} \mathrm{Im}(\gamma_n(w)) \leq M_n$ for all n, where each $M_n > 0$, and such that $\Phi'_{I_{\gamma_n}(z,\epsilon)}(0) \to \Phi'_{I_\gamma(z,\epsilon)}(0)$ as $n \to \infty$ [Lawler2008],[LSW2003].

We will now proceed to show that the same result holds for curves which are constrained to lie within a given simply connected domain D. For such a simply connected domain D and $z, w \in \partial D$, let $\mathcal{X}_D$ be the set of all smooth $\gamma : [a,b] \to \overline{D}$ such that $\gamma(a,b) \subset D$, $\gamma(a), \gamma(b) \in \partial D$ and $\gamma[a,b]$ seperates D into two simply connected components seperating z and w. Denote these two connected components by D_1 and D_2, where $z \in \partial D_1$ and $w \in \partial D_2$. Let $\mathcal{X}_D^*$ denote the set of all $\gamma \in \mathcal{X}_D$ which are parametrized by arc length. If K is a restriction hull under the law $\mathbb{P}_\alpha^{(D,z,w)}$, we will say that $v \in D$ is a *generalized bridge point on* D if $K \cap \gamma[a,b] = \{v\}$. Let $C_{\gamma,D}$ denote the (random) set of generalized bridge points on D. In this case, we must concede that $C_{\gamma,D}$ be either the empty set or a singleton. For our purposes, however, this suffices.

Corollary 3.2.9. *Let D be a simply connected domain, $z, w \in \partial D$, and let $\gamma \in \mathcal{X}_D^*$. Then if K is a restriction hull under the law $\mathbb{P}_\alpha^{(D,z,w)}$, conditioning on the event that $v \in C_{\gamma,D}$ and considering the hull K up until the first time it hits $\gamma[a,b]$, one obtains the law for a restriction hull $\hat{K}$ on the triple (D_1, z, v).*

Proof. Let $f : D \to \mathbb{H}$ be a conformal transformation with $f(z) = 0$, $f(w) = \infty$. Let $A \subset D_1$ be such that $A = \overline{A \cap D_1}$, $D_1 \setminus A$ is simply connected and bounded away from z, v. As before we define $I_\gamma(v, \epsilon) := \gamma[a, s_v - \epsilon] \cup \gamma[s_v + \epsilon, b]$, where s_v is such that $\gamma(s_v) = v$. We define the conditioning $\mathbb{P}_\alpha^{(D,z,w)}\{\cdot | v \in C_{\gamma,D}\}$ as

$$\mathbb{P}_\alpha^{(D,z,w)}\{\cdot | v \in C_{\gamma,D}\} := \lim_{\epsilon \to 0+} \mathbb{P}_\alpha^{(D,z,w)}\{\cdot | K \cap I_\gamma(v,\epsilon) = \emptyset\}.$$

Now by conformal invariance,

$$\begin{aligned}
\mathbb{P}_\alpha^{(D,z,w)}\{K \cap A = \emptyset | v \in C_{\gamma,D}\} &= \lim_{\epsilon \to 0+} \mathbb{P}_\alpha^{(D,z,w)}\{K \cap A = \emptyset | K \cap I_\gamma(v,\epsilon) = \emptyset\} \\
&= \lim_{\epsilon \to 0+} \mathbb{P}_\alpha\{\tilde{K} \cap f(A) = \emptyset | \tilde{K} \cap f(I_\gamma(v,\epsilon)) = \emptyset\} \\
&= \mathbb{P}_\alpha\{\tilde{K} \cap f(A) = \emptyset | f(v) \in C_{f(\gamma)}\} \\
&= \mathbb{P}_\alpha^{(f(D_1),0,f(v))}\{\hat{\tilde{K}} \cap f(A) = \emptyset\} \\
&= \mathbb{P}_\alpha^{(D_1,z,v)}\{\hat{K} \cap A = \emptyset\}.
\end{aligned}$$

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Chapter 4
The infinite quarter-plane SAW

Here we conjecture the existence of an object called the *infinite length quarter-plane SAW* on the lattice $\mathbb{Z} = \mathbb{Z} + i\mathbb{Z}$. This is a probability measure on the space of SAWs ω such that $\omega_0 = 0$, and ω stays in the quarter-plane Re $z \geq 0$, Im $z \geq 0$. The construction is similar to that of the infinite upper half-plane SAW. The basic requirements for such a construction are a notion of bridges and irreducibility, and certain ratio limit theorems which can be derived from the pattern theorem.

The reason that the existence of the infinite quarter-plane SAW remains conjecture is that a proof of Conjecture 4.1.19 remains elusive. Every other detail of the construction has been seen to with full rigor. That is to say, if one takes Conjecture 4.1.19 as a given, then the full existence of the infinite quarter-plane SAW follows immediately from what has been done here.

We begin by just stating some definitions and introducing the notations which will be used for the remainder of the chapter.

Definition 4.0.1. As in Section 2.1, the number of N-step SAWs starting at the origin will be denoted c_N. The number of N-step half-plane bridges starting at the origin will be denoted b_N, and the number of N-step half-plane SAWs starting at the origin will be h_N. The set of all N-step SAWs beginning at the origin will be denoted $\mathcal{S}_N$, while the set of N-step half-plane SAWs and half-plane bridges will be respectively denoted by $\mathcal{H}_N$, $\mathcal{B}_N$. An N-step irreducible bridge is a bridge which cannot be written as the concatenation of 2 or more (non-trivial) bridges. We will denote the number of N-step irreducible bridges beginning at the origin by λ_N, and the set of all such irreducible bridges by Λ_N.

Now we need to define something called a quarter-plane walk, which is similar to a half-plane walk, although the definition requires us to allow the walk to touch the real and imaginary axis at any point along the walk.

4.1 Bridges and the connective constant

Definition 4.1.1. An N-step quarter-plane self-avoiding walk ω in $\mathbb{Z}^2$, beginning at 0, is defined to be a SAW whose components satisfy the following inequalities:

- $\mathrm{Re}(\omega_0) \le \mathrm{Re}(\omega_j)$ for all $j = 1, \ldots, N$,
- $\mathrm{Im}(\omega_0) \le \mathrm{Im}(\omega_j)$ for all $j = 1, \ldots, N$.

The number of N-step quarter-plane SAWs beginning at the origin will be denoted q_N. The set of all N-step quarter-plane SAWs ω with $\omega_0 = 0$ will be denoted by $\mathcal{Q}_N$. By convention, we take $q_0 = 1$ (i.e. the trivial walk).

Let $\mathbb{Q}$ denote the closed quarter plane $\{z \in \mathbb{Z} : \mathrm{Re}(z), \mathrm{Im}(z) \ge 0\}$. This is somewhat of an abuse of notation, since in Chapter 5 we will use $\mathbb{Q}$ to denote the open quarter plane, $\{z \in \mathbb{H} : \mathrm{Re}(z) > 0\}$. Given a lattice site $z \in \mathbb{Z}^2$, let $\mathbb{Q}_z$ denote the quarter plane translated to z, i.e.

$$\mathbb{Q}_z := \mathbb{Q} + z.$$

Definition 4.1.2. An N-step *quarter-plane bridge* ω, beginning at 0, is defined to be an $\omega \in \mathcal{Q}_N$ such that there exists $z \in \mathbb{Z}^2$ satisfying

$$\partial \mathbb{Q}_z \cap \omega = \{\omega_N\} = \{z\}.$$

The number of N-step quarter-plane bridges ω beginning at 0 will be denoted ρ_N. By convention, $\rho_0 = 1$ (we think of this as the trivial walk, i.e. $\omega = \{0\}$). We will let $\mathcal{P}_N$ denote the set of N-step quarter-plane bridges ω beginning at 0, so that $\rho_N = |\mathcal{P}_N|$.

Note that the concatenation of two quarter-plane bridges is a quarter-plane bridge, and therefore we have

$$\rho_N \rho_M \leq \rho_{N+M},$$

which means that the sequence $(-\log \rho_N)$ is subadditive, and therefore the existence of the constant

$$\beta_{Qbridge} = \lim_{N\to\infty} \rho_N^{1/N} = \sup_{N\geq 1} \rho_N^{1/N}$$

is guaranteed.

Since $\rho_N \leq q_N$ for all N, this also gives us the existence of the constant $\beta_{Quarter}$, defined by $\beta_{Quarter} := \lim_{N\to\infty} q_N^{1/N}$.

Since $q_N < h_N < c_N$, we know that $\beta_{Quarter} \leq \beta$, and we would like to know that $\beta_{Quarter} = \beta$. We will prove something stronger. That is, we will show not only that $\beta_{Quarter} = \beta$, but that moreover, $\beta_{Qbridge} = \beta$. We will first state a theorem by Hardy and Ramanujan (1917):

Theorem 4.1.3. *For each integer $A \geq 1$, let $P_D(A)$ denote the number of partitions of A into distinct integers (i.e. the number of ways to write $A = A_1 + \cdots + A_k$ where $A_1 > \cdots > A_K$). Then*

$$\log P_D(A) \sim \pi \left(\frac{A}{3}\right)^{1/2} \quad \text{as } A \to \infty.$$

To begin, we will prove a proposition which bounds the number of quarter-plane SAWs in terms of the number of quarter-plane bridges. We will proceed to bound the number of half-plane SAWs in terms of the number of quarter-plane SAWs. We will use these bounds to show that $\beta_{Qbridge} = \beta_{Quarter} = \beta$. We first need a definition:

Definition 4.1.4. An N-step walk ω is called a *bridge in the weak sense*, or *weak bridge*, if the imaginary components satisfy

- $\mathrm{Im}(\omega_0) \leq \mathrm{Im}(\omega_j) \leq \mathrm{Im}(\omega_N)$.

The number of N-step weak bridges starting at the origin is denoted $\tilde{b}_N$ (as opposed to b_N, which is used to denote the number of N-step bridges beginning at the origin). A *weakly half-plane SAW* is defined similarly. Note that $b_N \le \tilde{b}_N \le c_N$, which implies that

$$\lim_{N\to\infty} \left(\tilde{b}_N\right)^{1/N} = \beta. \tag{4.1.5}$$

Definition 4.1.6. The *span* of a SAW $\omega \in \mathcal{S}_N$ is defined to be

$$\operatorname{span}(\omega) := \max_{0\le j\le N} \operatorname{Im}(\omega_j) - \min_{0\le j\le N} \operatorname{Im}(\omega_j).$$

We let $q_{N,A}$ denote the number of $\omega \in \mathcal{Q}_N$ with span A, and similarly $\rho_{N,A}$ denotes the number of $\omega \in \mathcal{P}_N$ with span A.

Proposition 4.1.7. *For every $N \ge 1$,*

$$q_N \le P_D\left(N\right)\rho_N, \tag{4.1.8}$$

where $P_D\left(N\right)$ is defined as in Theorem 4.1.3.

Proof. Let $N \ge 1$, and let $\omega \in \mathcal{Q}_N$. Let $n_0 = 0$, and for $j = 1, 2, \ldots$, recursively define $n_j\left(\omega\right)$ and $A_j\left(\omega\right)$ such that

$$A_j\left(\omega\right) = \max_{n_{j-1}\le k\le N} (-1)^j \left(\operatorname{Im}(\omega_{n_{j-1}}) - \operatorname{Im}(\omega_k)\right),$$

and $n_j\left(\omega\right)$ is the largest value of k for which this maximum is attained. The recusrion is stopped at the smallest integer J such that $n_J = N$. This means that $A_{J+1}\left(\omega\right)$ and $n_{J+1}\left(\omega\right)$ are not defined. Then $A_1\left(\omega\right)$ is the span of ω (as well as the height of ω). In general, $A_{j+1}\left(\omega\right)$ is the span of $\left[\omega_{n_j}, \ldots, \omega_N\right]$, which is either a weakly half plane walk (which stays in the quarter plane) or a reflection of one. Moreover, each of the subwalks $\left[\omega_{n_j}, \ldots, \omega_{n_{j+1}}\right]$ is a bridge in the weak sense, or the reflection of one. Also, we have $A_1 > A_2 > \cdots > A_J$.

Now, given a set of J decreasing positive integers $a_1 > \cdots > a_J$, let $\mathcal{Q}_N\left[a_1, \ldots, a_J\right]$ denote the set of all quarter-plane SAWs $\omega \in \mathcal{Q}_N$ with $A_1\left(\omega\right) = a_1, \ldots, A_J\left(\omega\right) = a_J$

and $n_J(\omega) = N$. Note then that $\mathcal{Q}_N[a]$ is the set of all weak bridges of span a beginning at the origin which are also quarter-plane walks. Since each of these walks is a quarter-plane bridge, but not every quarter-plane bridge is a weak bridge, we have

$$|\mathcal{Q}_N[a]| \leq \rho_{N,a}, \tag{4.1.9}$$

Then, given $\omega \in \mathcal{Q}_N$, we define a new walk $\tilde{\omega}$ as follows. For $0 \leq j \leq n_1(\omega)$, let $\tilde{\omega}_j = \omega_j$ and for $n_1(\omega) < j \leq N$, let $\tilde{\omega}_j$ be the reflection of ω_j about the line $\mathrm{Im}(z) = A_1(\omega)$. Then $\tilde{\omega} \in \mathcal{Q}_N[a_1 + a_2, \ldots, a_J]$, and since this transformation is one-to-one, and since we can repeat the process until we end up with a weak bridge, we have

$$\begin{aligned} |\mathcal{Q}_N[a_1, \ldots, a_J]| &\leq |\mathcal{Q}_N[a_1 + a_2, \ldots, a_J]| \\ &\leq |\mathcal{Q}_N[a_1 + \cdots + a_J]|. \end{aligned}$$

Now, summing over all finite collections of decreasing integers $a_1 > \cdots > a_J$ and using (4.1.9), we have

$$\begin{aligned} q_N &= \sum |\mathcal{Q}_N[a_1, \ldots, a_J]| \\ &\leq \sum |Q_N[a_1 + \cdots + a_J]| \\ &\leq \sum \rho_{N,a_1+\cdots+a_J}, \end{aligned}$$

which in turn gives us that

$$\begin{aligned} q_N &\leq \sum_{A=1}^{\infty} P_D(A)\, \rho_{N,A} \\ &\leq P_D(N) \sum_{A=1}^{\infty} \rho_{N,A} \\ &= P_D(N)\, \rho_N, \end{aligned}$$

as desired. $\square$

We now proceed to state and prove a Theorem which will allow us to conclude one of the major necessary results required in the construction of the infinite quarter-plane SAW; we will show that

$$\beta_{Quarter} = \beta.$$

follows as a Corollary to the following Theorem.

Theorem 4.1.10. *For any constant $B > \pi\left(1/3\right)^{1/2}$, there exists an $N_0\left(B\right)$ such that*

$$h_N \leq q_N e^{BN^{1/2}} \text{ for all } N \geq N_0. \tag{4.1.11}$$

The proof that we present for this theorem relies on the following *unfolding process* depicted in Figure 4.1, which transforms half-plane SAWs into quarter-plane SAWs.

Proof. We first fix $B > \pi\left(1/3\right)^{1/2}$ and choose $\epsilon > 0$ such that $B - \epsilon > \pi\left(1/3\right)^{1/2}$. Then by Theorem 4.1.3, there exists a positive constant K' such that

$$P_D\left(A\right) \leq K' \exp\left[\left(B-\epsilon\right)A^{1/2}\right] \text{ for all } A. \tag{4.1.12}$$

Let $\omega \in \mathcal{H}_n$. Let $\tilde{A}_1 = \min_k \mathrm{Re}(\omega_k)$ and let $\tilde{m}_1$ be the first k for which $\mathrm{Re}(\omega_k) = \tilde{A}_1$. Notice that if $\tilde{A}_1 = 0$, then we have $\omega \in \mathcal{Q}_n$. Then we let

$$\tilde{B}_1\left(\omega\right) = \max_{0\leq k\leq \tilde{m}_1(\omega)} \mathrm{Re}(\omega_k)$$

and $\tilde{n}_1\left(\omega\right)$ is the first k for which $\mathrm{Re}(\omega_k) = \tilde{B}_1$. We then recursively define, for $j = 2, \ldots N$

$$\tilde{A}_j\left(\omega\right) = \min_{0\leq k\leq \tilde{n}_{j-1}} \mathrm{Re}(\omega_k)$$

and $\tilde{m}_j\left(\omega\right)$ is the first k such that $\mathrm{Re}(\omega_k) = \tilde{A}_j$. Similarly,

$$\tilde{B}_j\left(\omega\right) = \max_{0\leq k\leq \tilde{m}_j} \mathrm{Re}(\omega_k)$$

and $\tilde{n}_j\left(\omega\right)$ is the first k for which $\mathrm{Re}(\omega_k) = \tilde{B}_j$. The recursion stops for the first J at which either $\tilde{A}_{J+1}\left(\omega\right) = 0$ or $\tilde{B}_J\left(\omega\right) = 0$. If the recursion stops when $\tilde{A}_{J+1} = 0$, then

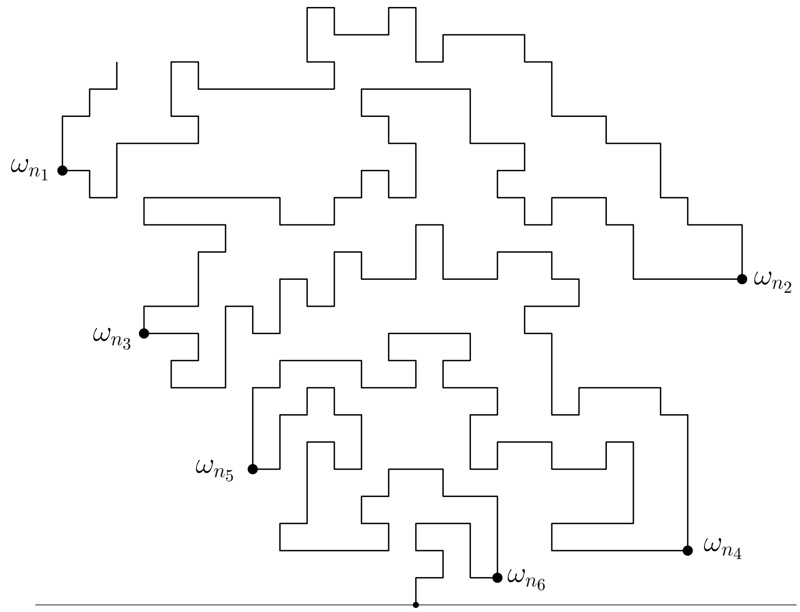

FIGURE 4.1. A SAW $\omega \in \mathcal{H}_{262}[256, 204, 148, 92, 45, 10]$. The sequence of folding times are marked. Here, the recursion stops at n_7, which we take to be undefined.

$\tilde{A}_{J+1}$ and $\tilde{B}_{J+1}$ are of thought of as undefined. If the recursion stops when $\tilde{B}_J = 0$, then both $\tilde{B}_J$ and $\tilde{A}_{J+1}$ are thought of as undefined. One might notice that the case where A_1 and B_1 are undefined is a special case (that is, ω is already a quarter-plane SAW), which will have to be taken into consideration in our calculations. We then define the sequence of positive integers $n_j(\omega)$ to be the *folding times* $n_1(\omega) = \tilde{m}_1(\omega)$, $n_2(\omega) = \tilde{m}_1(\omega)$, $n_3(\omega) = \tilde{m}_2(\omega)$, etc. In general, we define $n_{2j}(\omega) = \tilde{n}_j(\omega)$ and $n_{2j-1}(\omega) = \tilde{m}_j(\omega)$, where either $j = 1, \ldots, 2J$ or $j = 1, \ldots, 2J-1$.

Now, given a positive integer J and a sequence of J positive integers $a_1 > a_2 > \cdots > a_J > 0$, let $\mathcal{H}_N[a_1, \ldots, a_J]$ be the set of all $\omega \in \mathcal{H}_N$ such that $\omega_0 = 0$, $n_1(\omega) = a_1, \ldots, n_J(\omega) = a_k$ (hence n_{J+1} is undefined) . Note that for a given integer $a > 0$, $\mathcal{H}_N[a]$ is the set of all quarter plane SAWs reflected across the imaginary axis with $\omega_0 = 0$ and a the smallest integer such that $\mathrm{Re}(\omega_a) = \min_k \mathrm{Re}(\omega_k)$.

Given an N-step SAW ω in $\mathcal{H}_N[a_1, \ldots, a_J]$, define a new SAW $\omega^{[1]}$ as follows: First we define a translation of $\omega^{[1]}$, which we denote by $\tilde{\omega}^{[1]}$; For $j > a_1$, define $\tilde{\omega}^{[1]}_j = \omega_j$.

For $j \leq a_1$, define $\tilde{\omega}_j^{[1]}$ to be the reflection of ω_j about the line $\mathrm{Re}(z) = \mathrm{Re}(\omega_{a_1})$. We then define $\omega^{[1]}$ to be the horizontal translation of $\tilde{\omega}^{[1]}$ such that $\omega_0^{[1]} = 0$. Repeating this process J times, we see that $\omega^{[J]}$ is in $\mathcal{Q}_N$. Furthermore, this transformation is one-to-one, since given $\omega \in \mathcal{Q}_N$ which can be obtained through a sequence of such transformations from a half-plane SAW, and the sequence of folding times $(a_1, \ldots, a_J)$, $a_1 > \cdots > a_J > 0$, we can easily reproduce the half-plane SAW as follows: define a new SAW $\omega^{(1)}$ by first defining $\tilde{\omega}^{(1)}$ by $\tilde{\omega}_k^{(1)} = \omega_k$ for $k > a_J$ and by defining $\tilde{\omega}_k^{(1)}$ to be the relfection of ω_k across the line $\mathrm{Re}(z) = \mathrm{Re}(\omega_{a_J})$ for $k \leq a_J$, and then defining $\omega^{(1)}$ to be the horizontal translation of $\tilde{\omega}^{(1)}$ such that $\omega_0^{(1)} = 0$. We then repeat this process, i.e. we define $\omega_k^{(j)} = \omega_k^{(j-1)}$ for $k \leq a_{J-j+1}$ and $\omega_k^{(j)}$ to be the reflection across the line $\mathrm{Re}(z) = \mathrm{Re}(\omega_{a_{J-j+1}}^{(j-1)})$. Then $\omega^{(k)} \in \mathcal{H}_N\left[a_1, \ldots, a_J\right]$, and we see that the unfolding process is a one to one transformation. Therefore, summing over all sequences of positive decreasing integers $a_1 > \cdots > a_J > 0$, we have

$$\begin{aligned}
h_N &= \sum \left|\mathcal{H}_N\left[a_1, \ldots, a_J\right]\right| + q_N \\
&\leq \sum_{A=1}^{N} P_D\left(A\right) q_N + q_N \\
&\leq P_D\left(N\right) \sum_{A=1}^{N} q_N + q_N \\
&= \left(N P_D\left(N\right) + 1\right) q_N.
\end{aligned}$$

In other words, what we have just shown is that there exists a constant $C > 0$ such that for sufficiently large N,

$$h_N \leq C N P_D\left(N\right) q_N \tag{4.1.13}$$

Now, combining with (4.1.12), we have (for sufficiently large N),

$$h_N \leq K N q_N \exp\left[\left(B - \epsilon\right) N^{1/2}\right]$$

Therefore, there exists $N_0\left(B\right)$ such that

$$h_N \leq q_N e^{B N^{1/2}} \text{ for all } N > N_0. \tag{4.1.14}$$

□

Corollary 4.1.15. $\beta_{Quarter} = \beta$.

Proof. This fact is immediate from (4.1.14), since (4.1.14) implies that for sufficiently large N,

$$h_N e^{-BN^{1/2}} \le q_N \le h_N,$$

and consequently,

$$\beta = \lim_{N\to\infty} q_N^{1/N}.$$

□

Corollary 4.1.16. $\beta_{Qbridge} = \beta$.

Proof. The inequality (4.1.8), combined with (4.1.12), tell us that for sufficiently large N, we have

$$q_N \le \rho_N e^{BN^{1/2}},$$

and following a similar argument to that in the proof of Corollary 4.1.15, we see that

$$\beta = \lim_{N\to\infty} \rho_N^{1/N}. \tag{4.1.17}$$

□

Definition 4.1.18. The generating function for the number of quarter-plane bridges is denoted $P(z)$, and is given by

$$P(z) = \sum_{\omega\in\mathcal{P}} z^{|\omega|} = \sum_{N\ge 0} \rho_N z^N,$$

where $\mathcal{P} = \bigcup_{N=0}^{\infty} \mathcal{P}_N$ is the set of all quarter-plane bridges ω with $\omega_0 = 0$ of any length.

Notice that equation (4.1.17) tells us that the radius of convergence of $P(z)$ is $z_c = \beta^{-1}$. We will now state the conjecture that the rest of the construction hinges upon.

As mentioned previously, a proof of the conjecture remains elusive, though we believe it to be true. The natural thing to attempt might be to try to recreate the proof of Proposition 2.1.20 in this context, though it appears that there is no decomposition of quarter-plane SAWs into quarter-plane bridges analogous to that utilized in the proof of Proposition 2.1.20, nor is there any such one-to-one decomposition of half-plane SAWs into quarter-plane bridges. We leave it open as conjecture with the awareness that, should a proof be found, then the construction of the infinite quarter-plane SAW would be complete.

Conjecture 4.1.19.

$$\lim_{z \nearrow z_c} P(z) = +\infty;$$

that is,

$$P(z_c) = \sum_{\omega \in \mathcal{P}} \beta^{-|\omega|} = \sum_{N=1}^{\infty} \rho_N \beta^{-N} = +\infty. \tag{4.1.20}$$

4.2 Irreducible bridges and a renewal equation.

Definition 4.2.1. Given $\omega \in \mathcal{P}_N$, we will say that ω is a *reducible* bridge if there exist $\tilde{\omega} \in \mathcal{P}_n$, $\omega' \in \mathcal{P}_m$, $n + m = N$, $n, m \neq 0$ such that $\omega = \tilde{\omega} \oplus \omega'$. A quarter-plane bridge which is not reducible will be called *irreducible.*

We will let Υ_N denote the set of all irreducible quarter-plane bridges beginning at the origin, and $\upsilon_N := |\Upsilon_N|$. It is going to be useful to consider the generating function for irreducible quarter-plane bridges:

Definition 4.2.2. The generating function, $\Upsilon(z)$, for irreducible quarter-plane bridges, is defined to be

$$\Upsilon(z) = \sum_{\omega \in \Upsilon} z^{|\omega|} = \sum_{N=1}^{\infty} \upsilon_N z^N,$$

where the first sum is taken over the set of all irreducible quarter-plane bridges, $\Upsilon = \bigcup_{N=1}^{\infty} \Upsilon_N$.

Proposition 4.2.3.

$$\lim_{z \nearrow z_c} \Upsilon(z_c) = 1,$$

where, as above, $z_c = \beta^{-1}$. In other words,

$$\sum_{\omega \in \Upsilon} \beta^{-|\omega|} = \sum_{N=1}^{\infty} \upsilon_N \beta^{-N} = 1. \tag{4.2.4}$$

Proof. The proof of this proposition is the same as the proof of Proposition 2.1.20 , and essentially only relies on the notion of irreducibility, along with equation (4.1.20). We will prove it in full detail for the sake of clarity. The idea is as follows: Given $\omega \in \mathcal{P}_N$, let $s(\omega)$ be the smallest integer j $(1 \leq j \leq N)$ such that

$$\omega = \tilde{\omega} \oplus \omega',$$

where $\tilde{\omega} = [\omega_0, \ldots, \omega_j] \in \mathcal{P}_j$, $\omega' = [0, \omega_{j+1} - \omega_j, \ldots, \omega_N - \omega_j] \in \mathcal{P}_{N-j}$. Then, by the minimality of s, we see that $\tilde{\omega} \in \Upsilon_N$ and it follows that

$$\rho_N = \sum_{s=1}^{N} \upsilon_s \rho_{N-s} + \delta_{N,0}. \tag{4.2.5}$$

Multiplying both sides by z^N, and then summing N from 0 to ∞ yields

$$Q(z) = \Upsilon(z) Q(z) + 1,$$

and thus

$$Q(z) = \frac{1}{1 - \Upsilon(z)}. \tag{4.2.6}$$

Now, since the radius of convergence of $Q(z)$ is $z_c = \beta^{-1}$, we see that for $0 \leq z < z_c$, we must have $0 \leq \Upsilon(z) < 1$, and by (4.1.20), we immediately see that

$$\lim_{z \to z_c -} \Upsilon(z_c) = 1, \tag{4.2.7}$$

as desired. $\square$

We will most commonly write the result (4.2.7) as

$$\sum_{\omega \in \Upsilon} \beta^{-|\omega|} = 1,$$

or

$$\sum_{N=1}^{\infty} \upsilon_N \beta^{-N} = 1. \tag{4.2.8}$$

It is not difficult to see that equation (4.2.8) gives rise to a renewal equation. In fact, it is a standard result for irreducible half-plane bridges, and the same result holds for irreducible quarter-plane bridges. If we let (for $k \geq 1$)

$$p_k = \upsilon_k \beta^{-k}$$

and

$$a_N = \rho_N \beta^{-N},$$

for $N \geq 1$, then multiplying both sides of (4.2.5)by β^{-N} yields

$$a_N = \sum_{k=1}^{N} p_k a_{N-k}. \tag{4.2.9}$$

We can interpret this probabilistically as follows: Suppose we have a sequence of identically distributed random variables $X_1, X_2, \ldots$ on a probability space $(\mathbf{P}, \Omega)$ such that $\mathbf{P}\{X_j = k\} = p_k$ for every j. Then (4.2.9) tells us that

$$a_N = \mathbf{P}\{X_1 + \cdots + X_k = N \text{ for some } k \geq 0\}.$$

In other words, a_N is the probability that there is a "renewal" at "time" N. Equation (4.2.9) is called a renewal equation, and this probabilistic interpretation will fuel the construction of the infinite length quarter-plane SAW.

4.3 The pattern theorem and the ratio limit theorems

4.3.1 The Pattern Theorem

The final ingredient required in the construction of the infinite length quarter-plane SAW is going to be Kesten's pattern theorem, applied to quarter-plane SAWs. Briefly, a pattern is a short SAW which can occur as part of a longer SAW. Kesten's pattern theorem, as originally stated, says that if a given pattern can occur many times on a SAW, then the probability that the pattern occurs less than aN time on an N-step SAW, for some $a > 0$, decays exponentially in N as $N \to \infty$. Kesten's pattern theorem was originally stated and proved for full plane SAWs [Kesten1963], and has been checked to be valid for half-plane SAWs [LSW2002], half-plane bridges, and SAWs with a given initial point and terminal point [MS1993]. We will state and fully prove the theorem in the case of the quarter-plane SAWs. With this, we will be able to prove a version of Kesten's ratio limit theorems for quarter-plane SAWs, and this will be the final ingredient for our construction of the infinite length quarter-plane SAW.

Let us begin by defining explicitly what we mean by a pattern.

Definition 4.3.1. A *pattern* $P = [p_0, \ldots, p_n]$ is said to *occur at the j-th step of the SAW* $\omega = [\omega_0, \ldots, \omega_N]$ if there exists a vector $v \in \mathbb{Z}^2$ such that $\omega_{j+k} = p_k + v$ for $k = 0, \ldots, n$. It is clear that v must be defined by $v = \omega_j - p_0$.

But the story is slightly more complicated. The pattern theorem is not going to hold for general patterns, since these patterns can take on very pathological shapes. We must, in turn, make explicit what we mean when we say that a pattern can occur several times on a SAW. We will limit our attention to the case that $\omega \mathcal{Q}_N$.

Definition 4.3.2. For $k \geq 0$ and P a pattern, let $q_N [k, P]$ (respectively $\rho_N [k, P]$) denote the number of $\omega \in \mathcal{Q}_N$ (respectively, $\omega \in \mathcal{P}_N$) such that the pattern P occurs no more than k times on ω. Let $\mathcal{G}_N [P]$ (respectively $\mathcal{J}_N [P]$) denote the subset of $\mathcal{Q}_N$ (respectively $\mathcal{P}_N$) for which P occurs at the 0-th step. We will say that P is a proper front pattern if $\mathcal{G}_N [P]$ (respectively $\mathcal{J}_N [P]$) is nonempty for all sufficiently large N. We will not make any attempt to linguistically distinguish between proper front patterns for quarter-plane SAWs and proper front patterns for quarter-plane bridges. It should be clear from context which we shall be referring to. We will say that P is a proper front pattern if for all $k \geq 1$, there exists an $N \geq 1$ and a SAW $\omega \in \mathcal{Q}_N$ (respectively $\omega \in \mathcal{P}_N$) such that P occurs at least k times for ω. Once again, we will make no attempt to distinguish between proper internal patterns for $\omega \in \mathcal{Q}_N$ and $\omega \in \mathcal{P}_N$, and leave it up to context to distinguish to which we are referring.

With these definitions in hand, we can properly state Kesten's pattern theorem. The theorem states that if P is any proper internal pattern, then there exists an $a > 0$ such that

$$\limsup_{N\to\infty} \left(q_N\left[aN, P\right]\right)^{1/N} < \beta. \tag{4.3.3}$$

It is disputable whether it is clear that (4.3.3) gives you the probabilistic statement mentioned at the beginning of the section. To see this, observe that (4.3.3) implies the existence of an $\epsilon > 0$ such that

$$\limsup_{N\to\infty} \left(q_N\left[aN, P\right]\right)^{1/N} < \beta\left(1-\epsilon\right),$$

and therefore for all sufficiently large N, we have

$$q_N\left[aN, P\right] \leq \left(\beta\left(1-\epsilon\right)\right)^N.$$

Also, since $\beta = \lim_{N\to 1} q_N^{1/N}$, we see that there exists for all sufficiently large N,

$$q_N \leq \left(\beta\left(1+\epsilon\right)\right)^N.$$

Therefore, we see that the probability that a given N step quarter-plane SAW ω with $\omega_0 = 0$ has no more than aN occurences of the pattern P satisfies

$$\frac{q_N\left[aN, P\right]}{q_N} \leq \left(\frac{1-\epsilon}{1+\epsilon}\right)^N$$

for all sufficiently large N.

Let us now state some equivalent characterizations of proper internal patterns which will help us throughout the proof of the pattern theorem.

Proposition 4.3.4. *Let P be a pattern. The following are equivalent:*

(a) P is a proper internal pattern;

(b) There exists a cube $Q = \{x \in \mathbb{Z}^2 : 0 \leq x_i \leq b\}$ and a SAW ϕ such that: P occurs at some step of ϕ, ϕ is contained in Q, and the two endpoints of ϕ are corners of Q;

(c) There exists a SAW ω such that P occurs at three or more steps of ω.

One should note that here we are using the more vague terminology of SAW as opposed to being explicit about which type of SAW to which we are referring. We will use SAW here to refer to either quarter-plane SAWs or quarter-plane bridges. Also, in the definition of the cube $Q = \{x \in \mathbb{Z}^2 : 0 \leq x_i \leq b\}$, we are using the subscripts i to denote the real and imaginary parts of x; $x_1 = \mathrm{Re}(x)$, $x_2 = \mathrm{Im}(x)$.

It will also be useful to notice that if (b) above holds for a pattern P, then it will always be possible to take

$$b = 2 + \max\left\{||u - v||_\infty : \ u \text{ and } v \text{ are sites of P}\right\}.$$

The proof that (a) $\implies$ (c) is clear, as well as the proof that (b) $\implies$ (c) and (b) $\implies$ (a). It is more difficult to show that (c) implies both (a) and (b), though one can find a proof of this in Hammersley and Whittington.

We will now add some notation that will help us along in our quest to prove the pattern theorem.

Definition 4.3.5. A *cube* is any set of the form

$$Q = \left\{x \in \mathbb{Z}^2 : a_i \leq x_i \leq a_i + b \text{ for all } i = 1, 2\right\},$$

where a_1, a_2 and b are integers and $b > 0$. Each cube has $2^2 = 4$ corners. If Q is a cube as above, then let $\overline{Q}$ denote the cube which is two units larger in all directions;

$$\overline{Q} = \left\{x \in \mathbb{Z}^2 : a_i - 2 \leq x_i \leq a_i + b + 2 \text{ for all } i = 1, 2\right\};$$

and let ∂Q denote the set of points which are in $\overline{Q}$ but not in Q;

$$\partial Q = \overline{Q} \backslash Q.$$

An *outer point* of ∂Q is a point of ∂Q which has at least one nearest neighbour which is not in Q.

Definition 4.3.6. Suppose that Q is a cube and that P is an n-step pattern such that p_0 and p_n are corners of Q, and $p_i \in Q$ for all $i = 0, \ldots, n$ (in particular, P is a proper internal pattern). We say that (P, Q) occurs at the j-th step of the SAW ω if there exists a $v \in \mathbb{Z}^2$ such that $\omega_{j+k} = p_k + v$ for every $k = 0, \ldots, n$ and ω_i is not in Q for $i < j$ or for $i > j + n$. For every $k \geq 0$, let $q_N [k; (P, Q)]$ (respectively $\rho_N [k; (P, Q)]$) denote the number of $\omega \in \mathcal{Q}_N$ (respectively $\mathcal{P}_N$) for which (P, Q) occurs at no more than k different steps of ω.

We are now ready to state Kesten's Pattern Theorem in its full generality:

Theorem 4.3.7 (Pattern Theorem)**.** *(a) Let Q be a cube and P be a pattern as in definition 19. Then there exists an $a > 0$ such that*

$$\limsup_{N \to \infty} \left(q_N [aN, (P, Q)] \right)^{1/N} < \beta. \tag{4.3.8}$$

(b) For any proper internal pattern P, there exists an $a > 0$ such that

$$\limsup_{N \to \infty} \left(q_N [aN, P] \right)^{1/N} < \beta. \tag{4.3.9}$$

We will remark here that (4.3.8) is a stronger statement than (4.3.9). To see this, suppose that P is a proper internal pattern. Then there exists a ϕ and Q as in Proposition 4.3.4. Then, since P occurs on ϕ, any walk for which (ϕ, Q) occurs for m different steps must contain m or more occurences of P. Therefore, we have

$$q_N [k, P] \leq q_N [k, (\phi, Q)]$$

for every $k \geq 0$, which shows that (a) is a stronger statement than (b). Therefore, in order to prove Theorem 4.3.7, it suffices to prove part (a).

We will now prove a lemma which will serve as the first fundamental ingredient in our proof. In part (a), we will construct a pattern which completely fills a cube. In part (b), we will show that it is possible to splice a given pattern onto a SAW by first deleting the portion of the walk contained within the enlarged cube $\overline{Q}$, and then replacing that portion with the pattern we will construct in the lemma.

Lemma 4.3.10. *(a) Let Q be a cube in $\mathbb{Z}^2$. Then there exists a SAW ω, whose endpoints are corners of Q, which is entirely contained in Q and visits every point in Q (it should be noted here that the number of steps in ω is one less than the number of points in Q).*

(b) Let $P = [p_0, \ldots, p_k]$ be a pattern contained in the cube Q, whose endpoints are corners of Q. Let x and y be two distinct outer points of ∂Q. Then there exists a SAW ω' with the following properties: its initial point is x and its terminal point is y; it is entirely contained in Q; there is an integer j such that $\omega'_{j+i} = p_i$ for every $i = 0, \ldots, k$, $\omega_i \in \partial Q$ for every $i < j$ and every $i > j + k$. In particular, (P, Q) occurs at the j-th step of ω'.

We will prove the result for SAWs on the lattice $\mathbb{Z}^d$, as opposed to $\mathbb{Z}^2$. The reason for this is that the proof of part (a) relies on inducting on the dimension d, and although the proof will hold for $\mathbb{Z}^2$, there is no reason here to avoid a more general result for $\mathbb{Z}^d$ for this particular geometric lemma.

Proof. As stated above, we will prove (a) by induction on the dimension. For $d = 1$, the result is obvious. Suppose that the result holds for $d - 1$. For simplicity, assume that

$$Q = \left\{ x \in \mathbb{Z}^d : 0 \leq x_i \leq b \text{ for all } i = 1, \ldots, d \right\},$$

where b is a positive integer. Notice then that the intersection of Q with each of the hyperplanes $x_d = l$, $l = 0, \ldots, b$ is a $d-1$ dimensional cube, which we will call Q^l. By the inductive hypothesis, there exists a SAW $\omega^{[0]}$ which begins at the origin, whose endpoints are corners of Q^0, is entirely contained in Q^0 and which entirely fills up Q^0. Since each corner of Q^l is a nearest neighbour of Q^{l+1}, it follows that we can obtain the desired SAW by simply filling up each Q^l in turn.

(b) First we choose a SAW $\omega^{[1]}$ which goes from x to p_0 and contains only outer points of ∂Q (except for, necessarily, the last two steps). We then choose a SAW $\omega^{[2]}$ which goes from p_k to y and which is entirely contained in ∂Q (to do this, we simply avoid outer points until the very last step of $\omega^{[2]}$). Then the desired SAW is $\omega' = \omega^{[1]} \oplus P \oplus \omega^{[2]}$. □

We should remark here that in the proof of (a), choosing such a Q does not inhibit our ability to perform such an operation for quarter-plane SAWs. In fact, the SAW obtained in (a) is indeed a quarter-plane SAW. Similarly, for part (b), so long as the enlarged cube, $\overline{Q}$, is entirely contained in the quarter plane, we can perform this operation as many times as we would like for quarter-plane SAWs. This will be useful, because when we combine parts (a) and (b), we will be able to splice patterns which completely fill up cubes onto quarter-plane SAWs.

We will now need to expand our plethora of notation even further, creating some subtle ambiguities that can easily lead to some confusion. However, this seems necessary since it allows us to make arguments which would otherwise have to be repeated.

We begin by fixing a certain "radius," r, a positive integer. For a given N-step SAW (in particular, quarter-plane SAWs or quarter-plane bridges), ω, we will extend the definition of cube to that of cubes centered at some step of the walk ω. For $j = 0, \ldots, N$, let

$$\begin{aligned}
Q(j) &= \left\{x \in \mathbb{Z}^2 : |x_i - \pi_i \omega_j| \le r \text{ for all } i = 1, 2\right\}, \\
\overline{Q}(j) &= \left\{x \in \mathbb{Z}^2 : |x_i - \pi_i \omega_j| \le r + 2 \text{ for all } i = 1, 2\right\}, \\
\partial Q(j) &= \overline{Q}(j) \backslash Q(j)
\end{aligned}$$

where π_i denotes projection onto the i-th component. We will say that E^* occurs at the j-th step of ω if ω visits every site of the cube $Q(j)$. For a given integer $k \geq 1$, we will say that E_k occurs at the j-th step of ω if ω visits at least k sites of the enlarged cube $\overline{Q}(j)$. We will say that $\tilde{E}_k$ occurs at the j-th step of ω if either E^* or E_k (or both) occur there.

In what follows, we will use E to denote either E^*, E_k or $\tilde{E}_k$. If m is a positive integer, we say that $E(m)$ occurs at the j-th step of ω if E occurs at the m-th step of the $2m$-step SAW $[\omega_{j-m}, \ldots, \omega_j, \omega_{j+1}, \ldots, \omega_{j+m}]$. It is clear that this definition requires some modifying if $j - m < 0$ or $j + m > N$ (N the number of steps in ω). If $j - m < 0$, we say that $E(m)$ occurs at the j-th step of $[\omega_0, \ldots, \omega_{j+m}]$, and if $j+m > N$, we say that $E(m)$ occurs at the j-th step of ω if E occurs at the m-th step of $[\omega_{j-m}, \ldots, \omega_N]$. In particular, if $E(m)$ occurs at the j-th step of ω, then E occurs at the j-th step of ω. For every $k \geq 0$, let $q_N[k, E]$ (respectively, $q_N[k, E(m)]$denote the number of quarter-plane SAWs ω in Q_N such that E occurs at no more than k steps of ω. Note that for fixed k and N, $q_N[k, E(m)]$ is a non-increasing function of m, since occurences of $E(m)$ become more frequent as m increases.

The next lemma essentially just says that if E occurs on most quarter-plane SAWs, then $E(m)$ occurs quite often on most quarter-plane SAWs (quite often in the sense that the probability that it occurs less than aN times decays exponentially fast in N).

Lemma 4.3.11. *If*

$$\liminf_{N\to\infty} \left(q_N[0, E]\right)^{1/N} < \beta, \tag{4.3.12}$$

then there exists $a_1 > 0$ and an integer m such that

$$\limsup_{N\to\infty} \left(q_N[a_1 N, E(m)]\right)^{1/N} < \beta. \tag{4.3.13}$$

Proof. Observe first that $q_N[0, E] = q_N[0, E(N)]$, so there exists an $\epsilon > 0$ and a positive integer m such that

$$q_m [0, E(m)] < (\beta (1-\epsilon))^m \tag{4.3.14}$$

and

$$q_m < (\beta (1+\epsilon))^m . \tag{4.3.15}$$

To see this, observe that (4.3.12) implies that there exists an $\epsilon > 0$ such that $\inf_{n \geq N} (q_n [0, E(n)])^{1/n} < \beta (1-\epsilon)$ for all $N \geq 1$. Also, since $\beta = \lim_{N\to\infty} q_N^{1/N}$, there exists an integer $m_1 > 0$ such that $q_n < (\beta (1+\epsilon))^n$ for all $n \geq m_1$. then, since $\inf_{n \geq m_1} (q_n [0, E(m)])^{1/n} < \beta (1-\epsilon)$, it follows that there exists $m \geq m_1$ such that $q_m [0, E(m)] < (\beta (1-\epsilon))^m$, and thus we have established (4.3.14) and (4.3.15).

Now, let $\omega \in \mathcal{Q}_N$ and let $M = \llcorner N/m \lrcorner$. Now, for a given $0 \leq k \leq M$, if $E(m)$ occurs at most k times in ω, then $E(m)$ occurs in at most k of the M m-step subwalks $[\omega_{(i-1)m}, \omega_{(i-1)m+1}, \ldots, \omega_{im}]$, $i = 1, \ldots, M$. Crudely counting the number of ways in which $E(m)$ can occur at k or fewer of these subwalks (and remembering to count the last $N - mM$ steps of ω, we arrive at the (rough) bound

$$\begin{aligned} q_N [k, E(m)] &\leq \sum_{j=0}^{k} \binom{M}{j} q_m^j (q_m [0, E(m)])^{M-j} q_{N-mM} \\ &< \beta^{mM} q_{N-mM} \sum_{j=0}^{k} \binom{M}{j} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{jm} (1-\epsilon)^{mM} . \end{aligned} \tag{4.3.16}$$

Now, it suffices to show that there exists $\rho > 0$ and a positive $t < 1$ such that

$$q_N [\rho M, E(m)] < t \beta^m \tag{4.3.17}$$

for all sufficiently large M, since if that is true, for $0 < a_1 < \rho/m$, we have

$$q_N [a_1 N, E(m)]^{1/M} \leq q [\rho M, E(m)]^{1/M} < t \beta^m ,$$

for all sufficiently large M, (and thus, for all sufficiently large N), and by taking m-th roots, we obtain the existence of $\delta > 0$ such that

$$q_N [a_1 N, E(m)]^{1/N} < (1-\delta) \beta$$

for all sufficiently large N, which gives (4.3.15). But if ρ is a sufficiently small positive number, we obtain

$$\begin{aligned} &\textstyle\sum_{j=0}^{\rho M} \left(\begin{array}{c} M \\ j \end{array} \right) \left(\frac{1+\epsilon}{1-\epsilon}\right)^{jm} (1-\epsilon)^{Mm} \\ \leq \quad &(\rho M + 1) \left(\begin{array}{c} M \\ \rho M \end{array} \right) \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\rho M m} (1-\epsilon)^{Mm} . \end{aligned} \tag{4.3.18}$$

Here we use ρM on the sum instead of $\llcorner \rho M \lrcorner$ for readability. Using Stirling's formula, one can show that as $M \to \infty$, the M-th root of the right hand side of (4.3.18) converges to

$$\frac{1}{\rho^{\rho} (1-\rho)^{1-\rho}} \left(\frac{1+\epsilon}{1-\epsilon}\right)^{\rho m} (1-\epsilon)^{m} ,$$

which is less than 1 whenever $0 < \rho < \rho_0$ for some sufficiently small ρ_0. Therefore, by (4.3.16), we see that (4.3.17) holds whenever $0 < \rho < \rho_0$ and M is sufficiently large. □

It is worth remarking that the previous Lemma holds if we replace q_N with ρ_N wherever we see q_N. The initial argument for the existence of such an ϵ, m requires only a slight modification, but is essentially the same.

The next Lemma is referred to as the heart of the proof of the pattern Theorem. It is merely the statement that the hypothesis of Lemma 4.3.11 always holds in the case that E is E^*. That is, it says that almost all walks fill a cube of some fixed radius.

Lemma 4.3.19. $\liminf_{N\to\infty} q_N \left[0, E^*\right]^{1/N} < \beta.$

Proof. Begin by assuming that the lemma is false. That is, assume that

$$\lim_{N\to\infty} q_N \left[0, E^*\right]^{1/N} = \beta. \tag{4.3.20}$$

We must now make some very basic observations. First, notice that $q_N \left[0, \tilde{E}_k\right]$ is a nondecreasing function of k, since if $\tilde{E}_k$ does not occur on a given quarter-plane SAW, then $\tilde{E}_{k+1}$ surely does not. Also, if E^* does not occur on a given quarter-plane SAW, then it is clear that $\tilde{E}_{(2r+5)^2}$ cannot occur either (in order for $\tilde{E}_{(2r+5)^2}$ to occur, the enlarged cube, $\overline{Q}(j)$, must be entirely filled up, which cannot happen if $Q(j)$ never gets entirely filled up). These two observations lead us to the inequality

$$q_N[0, E^*] \leq q_N\left[0, \tilde{E}_{(2r+5)^2}\right] \leq q_N. \tag{4.3.21}$$

Thus, combining (4.3.21) with (4.3.20), and using the fact that $\lim_{N\to\infty} q_N^{1/N} = \beta$, we see that

$$\lim_{N\to\infty} q_N\left[0, \tilde{E}_{(2r+5)^2}\right]^{1/N} = \beta. \tag{4.3.22}$$

It is also not difficult to see that (for our fixed integer $r > 0$), we have $q_N\left[0, \tilde{E}_{r+3}\right] = 0$. This is because the first $r+3$ steps of any quarter-plane SAW must lie in $\overline{Q}(0)$. The fastest way for the walk to escape from $\overline{Q}(0)$ is to make $r+3$ consecutive steps in the vertical direction or in the horizontal direction. But this observation means that there must exist some constant K such that $r+3 \leq K < (2r+5)^2$ such that

$$\liminf_{N\to\infty} q_N\left[0, \tilde{E}_K\right]^{1/N} < \beta \tag{4.3.23}$$

and

$$\lim_{N\to\infty} q_N\left[0, \tilde{E}_{K+1}\right]^{1/N} = \beta. \tag{4.3.24}$$

Lemma 4.3.11, along with (4.3.23) then guarantee the existence of an $a_1 > 0$ and an integer m such that

$$\limsup_{N\to\infty} q_N\left[a_1 N, \tilde{E}_K(m)\right]^{1/N} < \beta. \tag{4.3.25}$$

Now, define the set T_N to be the set of quarter-plane SAWs $\omega \in \mathcal{Q}_N$ such that $\tilde{E}_K$ never occurs and such that $E_K(m)$ occurs at least $a_1 N$ times. That is,

$$T_N = \left\{\omega \in Q_N : \tilde{E}_{K+1} \text{ never occurs, } E_K(m) \text{ occurs at least } a_1 N \text{ times}\right\}. \tag{4.3.26}$$

In (4.3.26), we were able to replace $\tilde{E}_K(m)$ with $E_K(m)$, because requiring that $\tilde{E}_{K+1}$ cannot occur forces E^* to not occur. Now, observe that the number of quarter-plane SAWs in T_N must satisfy

$$|T_N| \geq q_N\left[0, \tilde{E}_{K+1}\right] - q_N\left[a_1 N, \tilde{E}_K(m)\right]. \tag{4.3.27}$$

This is easy to see. When substracting the number of $\omega \in Q_N$ such that $\tilde{E}_K(m)$ occurs no more than $a_1 N$ times, we must also count those same walks for which $\tilde{E}_{K+1}$ occurs, and therefore the number of walks in T_N exceeds this difference. Then, by (4.3.23), (4.3.24) and (4.3.27), we see that

$$\lim_{N\to\infty} |T_N|^{1/N} = \beta. \tag{4.3.28}$$

We can interpret this result to mean that there is some number K such that it is not unusual to find cubes with exactly K points covered and absolutely no cubes with $K+1$ points covered. For the rest of the proof, we will utilize Lemma 4.3.10, which states that we can splice patterns into quarter-plane SAWs by deleting enlarged cubes and replacing them with patterns which cover an entire cube. Given a walk $\omega \in T_N$, we will consider the collection of all cubes that have exactly K points covered. We will then proceed to remove the patterns inside a particular subcollection of these cubes and replace them with patterns which entirely fill the same cubes. This transformation is not one to one, and the length of the resulting walk will not be the same, but we can still arrange it so that the number of resulting walks is bigger than $|T_N|$ by an exponential factor, and this will contradict (4.3.20).

Suppose that $\omega \in \mathcal{Q}_N$ is such that $\tilde{E}_{K+1}$ never occurs and such that $E_K(m)$ occurs at the j_1-th, j_2-th, …, j_s-th steps of ω (here we allow for $E_K(m)$ to occur at more than these steps, but we assume that $E_K(m)$ occurs at at least these steps). Furthermore, assume that

$$0 < j_1 - m, j_s + m < N, \text{ and } j_l + m < j_{l+1} - m \text{ for all } l = 1, \ldots, s-1 \tag{4.3.29}$$

and

$$\overline{Q}(j_1), \ldots, \overline{Q}(j_s) \text{ are pairwise disjoint.} \tag{4.3.30}$$

The first condition (4.3.29) tells us that the first occurence of $E_K(m)$ we consider occurs after m steps and that the last occurence of $E_K(m)$ we consider occurs at least m steps before the end of the walk. This allows us to not have to deal with the different cases in which the definition of $E_K(m)$ would have to be modified. The second condition of (4.3.29) tells us that the bounded intervals in which each $E_K(m)$ occurs are completely seperated. Condition (4.3.30) will allow us to perform our splicing operation without running into any technical difficulties.

For $l = 1, \ldots, s$, let

$$\sigma_l = \min\left\{i : \omega_i \in \overline{Q}(j_l)\right\} \text{ and } \tau_l = \max\left\{i : \omega_i = \overline{Q}(j_l)\right\}$$

be the entrance and exit times, respectively, of ω in the enlarged cube $\overline{Q}(j_l)$. By our construction, since $E_K(m)$ occurs at the j_l-th step of ω and $\tilde{E}_{K+1}$ does not, this implies that there are exactly K sites of $\overline{Q}(j_l)$ covered by ω, and that these sites must lie between ω_{j_l-m} and ω_{j_l+m} on the walk. Therefore, we have $j_l - m \leq \sigma_l < j_l < \tau_l \leq j_l + m$ for every l. Consider all possible ways of replacing $[\omega_{\sigma_l}, \ldots, \omega_{\tau_l}]$ with a subwalk which stays completely inside of $\overline{Q}(j_l)$ and completely covers $Q(j_l)$. The existence of such subwalks is guaranteed by Lemma 4.3.10. We can perform this operation simultaneously for all subwalks $[\omega_{\sigma_l}, \ldots, \omega_{\tau_l}]$ (every $l = 1, \ldots, s$), since we have guaranteed that there be no overlap between subwalks and between the enlarged cubes. Furthermore, we can always choose $j_1, \ldots, j_s$ in such a way that the resulting SAW is a quarter-plane SAW ψ with $\psi_0 = 0$, and such that E^* occurs s times and such that the length, N', of ψ satisfies

$$N' < N + s(2r+5)^2. \tag{4.3.31}$$

This is easy to see, since $s(2r+5)^2$ is the total number of sites in $\overline{Q}(j_1), \ldots, \overline{Q}(j_s)$. Now, consider all triples (ω, ψ, J) where $\omega \in T_N$, $J = \{j_1, \ldots, j_s\}$ is some subset of $\{1, \ldots, N\}$ such that (4.3.29), (4.3.30) hold, $E_K(m)$ occurs at each $j_l \in J$, and $s = \llcorner \delta N \lrcorner$ (here δ is some small number to be defined later); and such that ψ can be obtained from ω by the above procedure. We shall provide rough estimates for the number of such triples from above and below in order to obtain a contradiction. In both cases, we will use the observation that each cube $\overline{Q}(j)$ intersects exactly

$V := (4r+9)^2$ cubes of radius $2+r$. This is because the cube $\overline{Q}(j)$ intersects the cube of radius r centered at x if and only if $||\omega(j)-x||_\infty \leq 2(r+2)$, and a simple count shows that there are exactly $(2(2(r+2))+1)^2 = (4r+9)^2$ such sites x.

First, the number of triples is at least the cardinality of T_N times the number of possible ways of choosing a set J for walks $\omega \in T_N$. Each $\omega \in T_N$ contains at least $a_1 N$ occurences of $E_K(m)$, so we may choose $h_1 < \cdots < h_u$, where $u = \llcorner a_1 N / (2m+2) V \lrcorner - 2$ such that *(i)* $E_K(m)$ occurs at the h_l-th step of ω for each $l = 1, \ldots, u$, *(ii)* $0 < h_1 - m, h_u + m < N$, $h_l + m < h_{l+1} - m$ for each $l = 1, \ldots, u-1$ and *(iii)* $\overline{Q}(h_1), \ldots \overline{Q}(h_u)$ are pairwise disjoint. This shows why we can choose this number for u; We divide $a_1 N$ by $(2m+2)V$ in the above expression to ensure no overlap between the subwalks in each of the $\overline{Q}(h_l)$, and we subtract 2 at the end in order to ensure that the subwalks don't occur at the beginning of the walk.

Then, it is clear that any subset of $\{h_1, \ldots, h_u\}$ which has cardinality $\llcorner \delta N \lrcorner$ is a possible choice for J. Therefore, we can see that if we set $\rho = a_1 / ((2m+2)V)$, then (dropping the $\llcorner \cdot \lrcorner$ from the notation), we obtain the bound

$$\text{number of triples } \geq |T_N| \begin{pmatrix} \rho N - 2 \\ \delta N \end{pmatrix}. \tag{4.3.32}$$

For an upper bound, consider a triple (ω, ψ, J). It is clear that E^* occurs at least $|J| = \llcorner \delta N \lrcorner$ times on ψ, though it may occur more than $\llcorner \delta N \lrcorner$ times since making a change in a cube $\overline{Q}(j_l)$ can produce occurences of E^* in some of the cubes of radius $r+2$ which intersect $\overline{Q}(j_l)$. But since E^* never occurs on ω, we see that E^* occurs at most $V|J|$ times on ψ. Therefore, given ψ, we see that there are at most $\begin{pmatrix} V\delta N \\ \delta N \end{pmatrix}$ possibilities for the locations of the cubes $\overline{Q}(j_l)$, $l = 1, \ldots, |J|$. Also, given ψ and the locations of the cubes $\overline{Q}(j_l)$, $l = 1, \ldots, |J|$, each such cube determines a subwalk of ψ that has replaced some subwalk of ω. Since each of the subwalks which were replaced by subwalks of ψ has at most $2m$ steps, we see that the total number of possibilites for ω, if we are given ψ and the locations of the cubes, is bounded from below by $\left(\sum_{i=0}^{2m} c_i\right)^{\delta N}$. Finally, if we know ω and the locations of the cubes, then J is uniquely determined. Defining $Z = \sum_{i=0}^{2m} c_i$, and using the elementary counting fact that $\left(\begin{array}{c} V\delta N \\ \delta N \end{array} \right) \leq 2^{V\delta N}$,along with (4.3.31), we see that

$$\text{number of triples } \leq 2^{V\delta N} Z^{\delta N} \sum_{i=0}^{N+(2r+5)^2\delta N} c_i. \tag{4.3.33}$$

This is clearly a very rough estimate, but it will suffice to give us the contradiction we desire. If we combine (4.3.31) and (4.3.32), take N-th roots and let $N \to \infty$, using Stirling's formula once again, along with (4.3.28), we see that

$$\beta \frac{\rho^\rho}{\delta^\delta (\rho - \delta)^{\rho-\delta}} \leq 2^{V\delta} \beta^{1+(2r+5)^2\delta} Z^\delta.$$

Thus, setting $Y = 2^V \beta^{(2r+5)^2} Z$ and $t = \delta/\rho$ gives

$$1 \leq \left(t^t (1-t)^{1-t} Y^t \right)^\rho.$$

We can see from elementary calculus that the function $f(t) = t^t (1-t)^{1-t} Y^t$ is less than 1 for sufficiently small $t > 0$, since $\lim_{t\to 0+} f(t) = 1$ and $\lim_{t \to 0+} f'(t) = -\infty$. Therefore, we obtain a contradiction and the Lemma is thus proved. □

We can now proceed to finally prove the pattern theorem. The final proof is essentially the same as the proof of lemma 23. All of the heavy lifting has essentially been done.

Proof of Theorem 4.3.7. . First, assume without loss of generality that the cube in the statement of the theorem is

$$Q = \left\{ x \in \mathbb{Z}^2 : |x_i| \leq r, i = 1, 2 \right\}.$$

As in the proof of Lemma 4.3.19, we will proceed by contradiction. That is, assume that the theorem is false; then for every $a > 0$,

$$\limsup_{N\to\infty} q_N \left[aN, (P, Q) \right]^{1/N} = \beta. \tag{4.3.34}$$

We introduce one more quick bit of notation. We shall say that E^{**} occurs at the j-th step of ω if the enlarged cube $\overline{Q}(j)$ is completely covered by ω. Now, by Lemmas

4.3.11 and 4.3.19 we obtain the existence of an $a' > 0$ and a positive integer m' such that

$$\limsup_{N\to\infty} q_N \left[a'N, E^{**}(m')\right]^{1/N} < \beta. \qquad (4.3.35)$$

Let $a > 0$ be some unspecified small number, and let H_N denote the following set of $\omega \in \mathcal{Q}_N$ such that (P, Q) occurs at most aN times on ω and $E^{**}(m')$ occurs at least $a'N$ times on ω. Then as in the proof of Lemma 4.3.19, the cardinality of H_N satisfies

$$|H_N| \geq q_N [aN, (P, Q)] - q_N [a'N, E^{**}(m')],$$

and thus, by (4.3.34) and (4.3.35), we have

$$\lim_{N\to\infty} |H_N|^{1/N} = \beta. \qquad (4.3.36)$$

Now, once again, let δ be a small positive number which will be specified at the end of the proof. This time, consider all triples (ω, v, J) such that: ω is in H_N; $J = \{j_1, \ldots, j_s\}$ is a subset of $\{1, \ldots, N\}$ such that $E^{**}(m')$ occurs at the j_l-th step of ω, and the conditions in (4.3.29) hold with m replaced by m', and $s = \llcorner \delta N \lrcorner$; and v is a quarter-plane SAW beginning at 0 that is obtained by replacing each occurrence of $E^{**}(m')$ by an occurence of (P, Q), analogously to the method described in the proof of Lemma 4.3.19. In this case, σ_l and τ_l are defined in the same way, and we use part (b) of Lemma 4.3.10 in order to be able to splice (P, Q) into the walk. Arguing in the same way as for (4.3.32), we see that

$$\text{number of triples } \geq |H_N| \binom{\rho N - 2}{\delta N}, \qquad (4.3.37)$$

where now $\rho = a'/(2m' + 2)$ (we need not multiply $(2m' + 2)$ by V, since we are not requiring that the enlarged cubes be disjoint. There is no need for this, since the occurence of $E^{**}(m')$ at each of the cubes ensures that the cubes be disjoint).

For the upper bound, we use the fact that v now has at most $aN + 2m'V\delta N$ occurences of (P, Q) (this allows for (i) at most aN occurences of (P, Q) on ω, and (ii) the maximum number of possibles ways that new occurences of (P, Q) could have been introduced, by either creating new occurences of (P, Q) or by vacating sites of other cubes. This is especially obvious if one sees that the number $2m'V\delta N$ can be thought of as follows: there are δN cubes onto which we splice an occurence of (P, Q); each one of the δN cubes is contained in a subwalk which has $2m'$ steps, and for each site in the original subwalk, the enlarged cube centered at that site intersets exactly V cubes of radius $r + 2$). It is also clear that since we are deleting portions of ω that fill an entire cube, then the resulting v has at most N steps. Therefore, the analogue of (4.3.33) is

$$\text{number of triples } \leq 2^{aN+2m'V\delta N} Z'^{\delta N} \sum_{i=0}^{N} c_i, \tag{4.3.38}$$

where $Z' = \sum_{i=0}^{2m'} c_i$. We now combine (4.3.37) and (4.3.38), put $a = \delta$, take N-th roots, let $N \to \infty$, and using Stirling's formula once again, by (4.3.36), we have

$$\beta \frac{\rho^\rho}{\delta^\delta (\rho - \delta)^{\rho-\delta}} \leq 2^{\delta+2m'V\delta} Z'^\delta \beta.$$

Once again, as in the proof of lemma 23, this leads to a contradiction for sufficiently small δ, and so the theorem is proven. □

4.3.2 The Ratio limit theorem

The pattern Theorem is perhaps the most fundamental result about many types of SAWs. However, it will not be used directly in our construction of the infinite length quarter-plane SAW. Instead, we use it as something of a preliminary result to prove the following ratio limit formula:

$$\lim_{N\to\infty} \frac{q_{N+2}}{q_N} = \beta^2. \tag{4.3.39}$$

(4.3.39) is the fundamental tool derived from the pattern theorem that will be used directly in the construction of the infinite length quarter-plane SAW. The following Lemma gives three conditions which together are sufficient for (4.3.39) to hold. The first two are very easy to verify, given what we have already done. The third is a little more complicated.

Lemma 4.3.40. *Let $\{a_N\}$ be a sequence of positive numbers and let $\phi_N = a_{N+2}/a_N$. Assume that*

(i) $\lim_{N\to\infty} a_N^{1/N} = \beta$,

(ii) $\liminf_{N\to\infty} \phi_N > 0$, *and*

(iii) there exists a constant $D > 0$ such that

$$\phi_N \phi_{N+2} \geq (\phi_N)^2 - \frac{D}{N}$$

for all sufficiently large N. Then

$$\lim_{N\to\infty} \phi_N = \beta^2.$$

This theorem is proved in full generality in Madras and Slade [MS1993], and is merely a statement about sequences of numbers. Therefore, we will skip the proof here and leave it to the interested reader to explore it in Madras and Slade.

It is clear that (i) holds in the case that $a_N = q_N$. Also, it is not hard to see that for sufficiently large N, we have

$$q_{N+2} \geq q_N,$$

which can be seen by taking an $\omega \in \mathcal{Q}_N$ and considering the last time it reaches its right most excursion. One can simply splice on a piece that extends it to the right by one unit and increases the length of the SAW by 2. This gives property (ii) in the case that $a_N = q_N$. It is left to show that property (iii) holds for $a_N = q_N$, and this is given by the following theorem:

Theorem 4.3.41. *There exists a constant $D > 0$ such that*

$$\phi_N \phi_{N+2} \geq (\phi_N)^2 - \frac{D}{N}$$

for all sufficiently large N, where ϕ_N is defined by $\phi_N = q_{N+2}/q_N$.

The proof of this is exactly the same as that of Theorem 7.3.2 in [MS1993].

4.4 The infinite quarter plane SAW

We will now construct the infinite quarter-plane SAW on $\mathbb{Z}^2$. To begin, we define the *uniform measure on* n-step quarter-plane SAWs $\omega \in \mathcal{Q}_N$ to be the measure $\mathbb{P}^n$ such that for any $\omega \in \mathcal{Q}_N$, we have

$$\mathbb{P}^n(\omega) = \frac{1}{q_n}.$$

Let $\mathcal{Q} := \bigcup_n \mathcal{Q}_n$, and extend $\mathbb{P}^n$ to Q by defining $\mathbb{P}^n(\omega) = 0$ whenever $\omega \in \mathcal{Q}_m$, $m \neq n$. As before, let $\Upsilon = \bigcup_n \Upsilon_n$ be the set of all irreducible quarter-plane bridges beginning at the origin. Then given $\omega^1, \ldots, \omega^l \in \Upsilon$, we define the cylinder set $\mathcal{Q}\left(\omega^1, \ldots, \omega^l\right)$ by

$$\mathcal{Q}\left(\omega^1, \ldots, \omega^l\right) = \left\{\omega \in \mathcal{Q} : \omega = \omega^1 \oplus \cdots \oplus \omega^l \oplus \tilde{\omega}, \tilde{\omega} \in \mathcal{Q}\right\}. \tag{4.4.1}$$

That is, $\mathcal{Q}\left(\omega^1, \ldots, \omega^l\right)$ is the set of all quarter-plane SAWs beginning at the origin which begin with the concatenation of the l irreducible quarter-plane bridges $\omega^1, \ldots, \omega^l$. If $|\omega^1| + \cdots + \left|\omega^l\right| = m$, then by the definition of the uniform measure, we have, for $n > m$,

$$\mathbb{P}^n\left(\mathcal{Q}\left(\omega^1, \ldots, \omega^l\right)\right) = \frac{q_{n-m}}{q_n}. \tag{4.4.2}$$

As in Section 2.1, we begin by constructing the measure which we will refer to as the infinite quarter-plane SAW. As in the case of the infinite half-plane SAW, our construction would be better named the infinite quarter-plane bridge. We will then show that this construction “makes sense” by showing the existence of the measure, $\mathbb{P}^\infty$, as the limit in distribution of the uniform measures $\mathbb{P}^n$ as $n \to \infty$. This is more aptly referred to as the infinite quarter-plane SAW, but we will show that it gives the same distribution as our constructed measure on infinite quarter-plane SAWs. In particular, we will prove that

$$\lim_{n\to\infty} \mathbb{P}^n\left(\mathcal{Q}\left(\omega^1, \ldots, \omega^l\right)\right) = \beta^{-m}, \tag{4.4.3}$$

and we will show how this can be used to construct the measure on infinite quarter-plane SAWs. It is worth noting that (4.4.3) would easily follow from (4.4.2) if we had the preliminary result

$$\lim_{n\to\infty}\frac{q_{n+1}}{q_n}=\beta.$$

To begin, observe that Conjecture 4.1.19 implies that $\Upsilon(z_c)=1$. Thus, a natural probability measure is induced on irreducible quarter-plane bridges which assigns weight $\beta^{-|\omega|}$ to each irreducible quarter-plane bridge ω starting at 0. Let ν_Υ denote this probability measure. Then, for $l\geq 1$, we define the probability measure ν_{Υ^l} on the cartesian product Υ^l according to product measure. Here, we identify the cartesian product Υ^l to the space of all l concatenated irreducible bridges, the first of which begins at 0, by $\left(\omega^1,\ldots,\omega^l\right)\leftrightarrow\omega^1\otimes\cdots\otimes\omega^l$, and thereby obtain a measure on the space of l concatenated irreducible quarter-plane bridges. To reiterate, given $\omega^1,\ldots,\omega^l\in\Upsilon$, we define

$$\begin{aligned}\nu_{\Upsilon^l}\left(\omega^1\oplus\cdots\oplus\omega^l\right) &= \nu_\Upsilon\left(\omega^1\right)\cdots\nu_\Upsilon\left(\omega^l\right) \\ &= \beta^{-\sum_{i=1}^l\left|\omega^i\right|}.\end{aligned}\tag{4.4.4}$$

We also write ν_{Υ^l} for the extension to $\mathcal{Q}$ given by $\nu_{\Upsilon^l}\left(\mathcal{Q}\backslash\Upsilon^l\right)=0$, and for the measure on $\mathcal{P}$ given by (4.4.4) whenever $\omega^1,\ldots,\omega^l\in\Upsilon$, and such that $\nu_{\Upsilon^l}(\omega)=0$ if $\omega\in\mathcal{P}$ is not of the form $\omega=\omega^1\oplus\cdots\oplus\omega^l$, $\omega^1,\ldots,\omega^l\in\Upsilon$.

Finally, we define ν_{Υ^∞} on $\Upsilon^\infty=\Upsilon\times\Upsilon\times\cdots$ according to the Kolmogorov consistency theorem, and it should be thought of as a measure on infinite self-avoiding paths (in fact, it is, a priori, a measure on infinite quarter-plane bridges). One should note that the consistency conditions are built into the definition of ν_{Υ^l}, and thus there are no problems in defining such a measure. Then, if $\mathcal{Q}^\infty\left(\omega^1,\ldots,\omega^l\right)$ denotes the set of all infinite length quarter-plane SAWs beginning with $\omega^1\oplus\cdots\oplus\omega^l$, and we have $|\omega^1|+\cdots\left|\omega^l\right|=m$, then

$$\nu_{\Upsilon^\infty}\left(\mathcal{Q}^\infty\left(\omega^1,\ldots,\omega^l\right)\right)=\beta^{-m},$$

and this is indeed a probability measure by (4.2.8).

We will show that the uniform probability measures $\mathbb{P}^n$ converge to ν_{Υ^∞} in distribution by proving (4.4.3). We will first need the following lemma:

Lemma 4.4.5.

$$\lim_{n\to\infty} \frac{q_{n+1}}{q_n} = \beta.$$

Proof. The proof follows that given in [LSW2002]. Given a quarter-plane SAW $\omega \in \mathcal{Q}_n$, let $s(\omega)$ denote the least renewal time; that is, $s(\omega)$ is the first k such that ω can be written as $\omega = \omega' \oplus \tilde{\omega}$, where $\omega' \in \Upsilon_k$ and $\tilde{\omega} \in \mathcal{Q}_{n-k}$. Then observe that

$$q_n = |\mathcal{Q}_n| \geq |\{\omega \in \mathcal{Q}_n : s(\omega) = m\}| = \sum_{k=1}^{m} \upsilon_k q_{n-k}.$$

Therefore, dividing by q_n, we have

$$1 \geq \sum_{0<k<m/2} \left(\upsilon_{2k} \frac{q_{n-2k}}{q_n} + \upsilon_{2k-1} \frac{q_{n-2k+1}}{q_{n-1}} \frac{q_{n-1}}{q_n} \right).$$

Now, taking the $\limsup$ as $n \to \infty$, we obtain

$$1 \geq \sum_{0<k<m/2} \left(\upsilon_{2k} \beta^{-2k} + \upsilon_{2k-1} \beta^{-2k+2} \limsup_{n\to\infty} \frac{q_{n-1}}{q_n} \right),$$

or

$$1 \geq \sum_{0<k<m/2} \left(\upsilon_{2k} \beta^{-2k} + \upsilon_{2k-1} \beta^{-2k+1} \limsup_{n\to\infty} \frac{\beta q_{n-1}}{q_n} \right).$$

Taking the limit $m \to \infty$, and using the fact (4.2.8), we arrive at

$$\sum_{k=1}^{\infty} \upsilon_k \beta^{-k} \geq \sum_{k=1}^{\infty} \left(\upsilon_{2k} \beta^{-2k} + \upsilon_{2k-1} \beta^{-2k+1} \limsup_{n\to\infty} \frac{\beta q_{n-1}}{q_n} \right).$$

Now, subtracting $\sum_{k=1}^{\infty} \upsilon_{2k} \beta^{-2k}$ from both sides, we have

$$\sum_{k=1}^{\infty} \upsilon_{2k-1} \beta^{-2k+1} \geq \sum_{k=1}^{\infty} \upsilon_{2k-1} \beta^{-2k+1} \limsup_{n\to\infty} \frac{\beta q_{n-1}}{q_n},$$

or equivalently,

$$0 \geq \sum_{k=1}^{\infty} \upsilon_{2k-1} \beta^{-2k+1} \left(\limsup_{n\to\infty} \frac{\beta q_{n-1}}{q_n} - 1 \right),$$

and since $\sum_{k=1}^{\infty} \upsilon_{2k-1}\beta^{-2k+1} > 0$, this implies that

$$\limsup_{n\to\infty} \frac{q_{n-1}}{q_n} \leq \beta^{-1}.$$

This together with (4.3.39) proves the lemma. □

Corollary 4.4.6. *If* $\omega^1, \ldots, \omega^l \in \Upsilon$ *with* $|\omega^1| + \cdots + |\omega^l| = m$*, then*

$$\lim_{n\to\infty} \mathbb{P}^n \left(\mathcal{Q}\left(\omega^1, \ldots, \omega^l\right)\right) = \beta^{-m}.$$

Proof. Take the limit as $n \to \infty$ in 4.4.2. □

If we denote $\mathbb{P}^\infty := \lim_{n\to\infty} \mathbb{P}^n$, where the limit is taken in distribution, as in the proof of Corollary 4.4.6, then the distributions of $\mathbb{P}^\infty$ and ν_{Υ^∞} coincide. This shows that the definition of the infinite quarter-plane SAW is a “good” definition.

Chapter 5
Restriction measures in the quarter plane

In chapter 4 we used $\mathbb{Q}$ denote the set of all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \geq 0$ and $\mathrm{Im}(z) \geq 0$. Here we will use the same symbol to denote the interior of this set. That is, let

$$\mathbb{Q} := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 : \mathrm{Im}(z) > 0\}.$$

Here we will show the existence of quarter plane bridges for restriction measures $\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}$ for $\alpha \in [5/8, 1)$. Recall that restriction measures are measures on restriction hulls, and that a restriction hull, K, is a stochastic process taking values in the space of unbounded hulls in $\mathbb{H}$ connecting 0 to ∞. An unbounded hull K connecting 0 to ∞ is a closed connected subset $K \subset \mathbb{C}$ such that $K \cap \overline{\mathbb{H}} = \{0\}$ and $\mathbb{H} \setminus K$ consists of exactly two connected components. Furthermore, we are able to parametrize these hulls in order to obtain a growing family of hulls $\{K_t\}$ such that $K_0 = \{0\}$ and $K_\infty = K$. In this section we will be considering restriction hulls on the triple $(\mathbb{Q}, 0, \infty)$, the law for which can be obtained by the law for restriction hulls on $(\mathbb{H}, 0, \infty)$ through conformal transformation. What we show in this section is a consequence of the restriction property and conformal invariance, which essentially characterize restriction measures (see Sections 2.3 and 3.1)

In [AC2010], it was shown that for restriction hulls under the law $\mathbb{P}_\alpha = \mathbb{P}_\alpha^{(\mathbb{H},0,\infty)}$, bridge points exist $\mathbb{P}_\alpha$-a.s. This was achieved by showing that the Hausdorff dimension for the set of bridge points of a restriction hull K under the law $\mathbb{P}_\alpha$ is equal to $2 - 2\alpha$. In this chapter we prove a similar result for restriction hulls on the triple $(\mathbb{Q}, 0, \infty)$. In section 5.1, we state the general Theorem, provide important definitions and prove that the Hausdorff dimension of the set of quarter-plane bridge points is constant $\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}$-a.s. In section 5.2, we prove that the Hausdorff dimension of the set of quarter-plane bridge points is equal to the maximum of $2 - 8/3\alpha$ and 0, $\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}$-a.s.

Note that the cutoff value of α for quarter-plane bridge points to exist is different than the cutoff value for half-plane bridge points. That is, in the half-plane, bridge points exist almost surely for $5/8 \leq \alpha < 1$. In the quarter, plane, bridge points cease to exist at the critical value $\alpha = 3/4$. In other words, quarter-plane bridge points exist for $5/8 \leq \alpha < 3/4$. The $\alpha = 3/4$ case remains unknown.

5.1 Definitions and initial machinery

Recall that we refer to the triple $(\mathbb{H}, 0, \infty)$ as the canonical triple. The law for a general triple (D, z, w) can then be obtained from the canonical triple through conformal invariance. Given $z \in \mathbb{Q}$, let $Q_z = \partial\mathbb{Q} + z$. If K is a restriction hull under the law $\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}$, we will say that $z \in \mathbb{Q}$ is a *quarter-plane bridge point* for K if $K \cap Q_z = \{z\}$. Let $\mathcal{C}$ denote the set of quarter-plane bridge points. We can then state our main theorem as follows:

Theorem 5.1.1. $\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}$*-almost surely, the Hausdorff dimension of* $\mathcal{C}$ *is equal to* $\max(2 - 8/3\alpha, 0)$.

Given a restriction hull K and $0 \leq t \leq s$, we define $\Lambda_{t,s}K$ to be the future hull between times t and s. We then define $\theta_{t,s}K = \Lambda_{t,s}K - \gamma(t)$, where $\gamma(t)$ is the SLE_κ curve generated by the hull K. Consider then the filtration $\mathcal{F}_t = \sigma(K_s : 0 \leq s \leq t)$. According to this filtration, the restriction measures satisfy the following domain Markov property:

$$\text{The conditional law of } \Lambda_t K \text{, given } \mathcal{F}_t \text{, is } \mathbb{P}_\alpha^{(\mathbb{H}\setminus\gamma[0,t],\gamma(t),\infty)},$$

where here $\Lambda_t K = \Lambda_{t,\infty}K$. Recall that restriction measures on the canonical triple satisfy the following restriction formula:

$$\mathbb{P}_\alpha\{K \cap A = \emptyset\} = \Phi_A'(0)^\alpha, \tag{5.1.2}$$

where A is a hull in $\mathbb{H}$ bounded away from 0 and Φ_A is a conformal map from $\mathbb{H} \setminus A$ onto A such that $\Phi_A(z) \sim z$ as $z \to \infty$. For general triples (D, z, w), the restriction formula is

$$\mathbb{P}_{\alpha}^{(D,z,w)}\{K \cap A = \emptyset\} = \Phi'_{f(A)}(0)^{\alpha}, \tag{5.1.3}$$

where A is a hull in D not containing z and w and f is a conformal map from D onto $\mathbb{H}$ such that $f(z) = 0$ and $f(w) = \infty$.

The proof of Theorem 5.1.1 relies on Proposition A.2.4. However, the one point and two point bounds alone are not enough to conclude the result of Theorem 6.2.1. In addition, we will need a 0-1 law, the argument of which will use the Blumenthall 0-1 law in order to conclude that the Hausdorff dimension of $\mathcal{C}$ is constant $\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}$-a.s.

Proposition 5.1.4. *The Hausdorff dimension of $\mathcal{C}$ is constant $\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}$-a.s.*

The proof of this theorem will mostly follow the method of the proof in [AC2010]

Proof. To begin, for $0 \le t \le s$, let $\mathcal{C}_t(s) = \{\text{quarter-plane bridge points of } K_s\} \cap K_t$ and for fixed $d > 0$, define the event $W_t(s) = \{\dim_H \mathcal{C}_t(s) \ge d\}$. It suffices to show that $\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{W_\infty(\infty)\} = 0$ or 1, for then there must exist some $d_0 \ge 0$ such that $\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{W_\infty(\infty)\} = 0$ for $d < d_0$ and $\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{W_\infty(\infty)\} = 1$ for $d \ge d_0$, from which one could conclude that $\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{\dim_H \mathcal{C} = d_0\} = 1$.

Note that $\mathcal{C}_t(s)$ is the set of bridge points of K_s that are also bridge points of K_t. Thus, it is easy to see that for fixed s, $\mathcal{C}_t(s)$ is increasing in t, and thus $W_t(s)$ as well. Similarly, for fixed t, increasing s only allows for the possibility of the future hull coming down and destroying bridge points of K_t, and therefore $\mathcal{C}_t(s)$ and $W_t(s)$ are decreasing in s for fixed t. For fixed s, set

$$V_s = \bigcap_{n=1}^{\infty} W_{\frac{s}{n}}(s) = \{\dim_H \mathcal{C}_t(s) \ge d \text{ for all } 0 < t \le s\}, \tag{5.1.5}$$

and then it follows that V_s is also decreasing in s. Now, for fixed s, consider $K \in V_s \setminus V_\infty$. Since $K \in V_s$, it follows that $\dim_H \mathcal{C}_t(s) \ge d$ for all $0 < t \le s$. Since $K \notin V_\infty$, it follows that there exists $t_0 \in (0,\infty)$ such that $\dim_H \mathcal{C}_{t_0}(\infty) < d$, and thus, since $\mathcal{C}_t(s)$ is increasing in t, it follows that $\dim_H \mathcal{C}_t(\infty) < d$ for all $t < t_0$. Notice then, that one can take such a t_0 to satisfy $0 < t_0 \leq s$, and it follows that for such t_0,

$$\dim_H \mathcal{C}_t(\infty) < d \leq \dim_H \mathcal{C}_t(s), \tag{5.1.6}$$

for all $0 < t \leq t_0$. But the only way that this can happen is if the future hull $\Lambda_s K$ wipes out bridge points of the hull K_t, and since this must be true for all $0 < t \leq t_0$, it follows that the future hull $\Lambda_s K$ must come arbitrarily close to either the real axis or the imaginary axis or both, all of which are measure zero events. Therefore, we have $\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}(V_s \setminus V_\infty) = 0$ for all $s > 0$, and since $\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}(V_s) = \mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}(V_\infty) + \mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}(V_s \setminus V_\infty)$, we have $\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}(V_s) = \mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}(V_\infty)$ for all s, and hence

$$\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}\left(\bigcap_{n=1}^{\infty} V_{\frac{1}{n}}\right) = \mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}(V_\infty). \tag{5.1.7}$$

It is clear that $\bigcap_{n=1}^{\infty} V_{\frac{1}{n}} \in \mathcal{F}_{0+}$, and therefore it simply remains to show that $\mathcal{F}_{0+}$ is independent of itself. In the case of $SLE_{8/3}$, this is true by the Blumenthal 0-1 law for Brownian motion, since $SLE_{8/3}$ is the pushforward of Weiner measure under the Loewner equation. For general $\alpha > 5/8$, the same property holds by a standard argument. Indeed, the Domain Markov property assures that $\Phi_{K_t}(\Lambda_t K)$ is a restriction hull independent of $\mathcal{F}_t$, and therefore if $A \in \mathcal{F}_{0+}$, it follows that for a bounded continuous function f on hulls (continuous with respect to the Carathedory topology), we have

$$\mathbf{E}_\alpha^{(\mathbb{Q},0,\infty)}\left[f\left(\Phi_{K_t}\left(\Lambda_t K\right)\right) 1_A\right] = \mathbf{E}_\alpha^{(\mathbb{Q},0,\infty)}\left[f\left(\Phi_{K_t}\left(\Lambda_t K\right)\right)\right] \mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}(A). \tag{5.1.8}$$

Then, taking a limit of both sides of (5.1.8) as $t \to 0+$, using the fact that f is continuous on hulls, that $K_t \to \{0\}$ so that Φ_{K_t} goes continuously to the identity, we get

$$\mathbf{E}_{\alpha}^{(\mathbb{Q},0,\infty)}\left[f\left(K\right)1_{A}\right]=\mathbf{E}_{\alpha}^{(\mathbb{Q},0,\infty)}\left[f\left(K\right)\right]\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\left(A\right), \tag{5.1.9}$$

which shows that A is independent of all elements of $\mathcal{F}_{\infty}$, and therefore of itself.

□

Now we use Proposition A.2.4 to prove Theorem 6.2.1. We define our thickened sets as follows:

Definition 5.1.10. For $z \in \mathbb{Q}$ and $\epsilon > 0$, let $J(z, \epsilon)$ be the set Q_z with an ϵ-corner removed. That is, let

$$J(z,\epsilon) := \{w \in \mathbb{Q} : \mathrm{Im}\, w = \mathrm{Im}\, z, \mathrm{Re}\,(w-z) \geq \epsilon\} \cup \{w \in \mathbb{Q} : \mathrm{Re}\, w = \mathrm{Re}\, z, \mathrm{Im}\,(w-z) \geq \epsilon\}. \tag{5.1.11}$$

Define the random set $\mathcal{C}_\epsilon$ by

$$\mathcal{C}_\epsilon = \{z \in \mathbb{Q} : K \cap J(z,\epsilon) = \emptyset\}. \tag{5.1.12}$$

With this definition, the following is true $\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}$-a.s.:

Proposition 5.1.13.

$$\bigcap_{\epsilon>0} \mathcal{C}_\epsilon = \mathcal{C} \tag{5.1.14}$$

Proof. By the definition of $\mathcal{C}$, $z \in \mathcal{C}$ if and only if $K \cap Q_z = \{z\}$. But this easily implies that $z \in \mathcal{C}_\epsilon$ for all $\epsilon > 0$. For the converse, suppose that $z \in \mathcal{C}_\epsilon$ for all $\epsilon > 0$. Then it is clear that z is the only possible element of $K \cap Q_z$. But the set $K \cap Q_\zeta$ is nonempty for all $\zeta \in \mathbb{Q}$ since restriction hulls are connected and connect 0 to ∞ $\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}$-a.s. Therefore, with $\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}$-probability 1, the set $K \cap Q_z = \{z\}$. □

5.2 Hausdorff measure of the set of quarter-plane bridge points

Now, to prove the one and two point bounds from Proposition A.2.4, we use the following version of the restriction formula (5.1.3): Given a hull $A \subset \overline{\mathbb{Q}}$ such that A is bounded away from 0, $\mathbb{Q} \setminus A$ is simply connected, let ϕ_A denote the conformal transformation $\Phi_{f(A)}$, where $f(z) = z^2$, and $\Phi_{f(A)}$ is the unique conformal transformation from $\mathbb{H} \setminus f(A)$ onto $\mathbb{H}$ such that $\Phi_{f(A)}(0) = 0$, $\Phi_{f(A)}(\infty) = \infty$ and $\Phi_{f(A)}(z) \sim z$ as $z \to \infty$.

$$\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\left(K \cap A = \emptyset\right) = \Phi_A'\left(0\right)^{\alpha}. \tag{5.2.1}$$

In fact, it can be shown (See [Lawler2008]) that formula (5.2.1) holds for unbounded A as well. Therefore, we can calculate the probability that, for a restriction hull in $\mathbb{Q}$, starting at 0 and ending at ∞, the point z is in the set $\mathcal{C}_\epsilon$ by the equation

$$\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{z \in \mathcal{C}_\epsilon\} = \mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{K \cap J(z,\epsilon) = \emptyset\} = \phi_{J(z,\epsilon)}'(0)^{\alpha}. \tag{5.2.2}$$

Similarly, we can get the two point estimate according to

$$\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{z, w \in \mathcal{C}_\epsilon\} = \mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{K \cap (J(z,\epsilon) \cup J(w,\epsilon)) = \emptyset\} = \phi_{J(z,\epsilon)\cup J(w,\epsilon)}'(0)^{\alpha}. \tag{5.2.3}$$

In order to calculate the above probabilities (5.2.2),(A.2.3), we will use the following result due to Bálint Virág [Virág2003]

Theorem 5.2.4. *Let D be a domain and let a, z be points on $\partial D \setminus A$ in a neighborhood of which the boundary is differentiable, where A is a hull in $\overline{D}$ not containing a, z. Let f be the conformal map which takes $D \setminus A$ to D and fixes a, z. Then*

$$\mathbf{P}\{\hat{B}(a,z,D) \text{ avoids } A\} = f'(a) f'(z), \tag{5.2.5}$$

where $\hat{B}(a,z,D)$ is the path of a Brownian excursion in D starting at a and ending at z.

We use the following version of this theorem to calculate $\phi'_{J(z,\epsilon)}(0)$:

Corollary 5.2.6. *Let $\hat{B}_t$ be a Brownian excursion in $\mathbb{Q}$ starting at 0 and ending at ∞, and let A be a compact quarter-plane hull; a set $A \subset \overline{\mathbb{Q}}$ such that $A = \overline{A \cap \mathbb{Q}}$, $\mathbb{Q} \setminus A$ is simply connected and A is bounded away from 0. Then*

$$\mathbb{P}\{\hat{B}[0,\infty) \cap A = \emptyset\} = \phi'_A(0). \tag{5.2.7}$$

We are now in a position to prove the following:

Proposition 5.2.8.

$$\phi'_{J(z,\epsilon)}(0) \asymp \epsilon^{8/3}, \tag{5.2.9}$$

Proof. We will prove the result for the case when $z = x + i$, $x \in \mathbb{R}$. We can then obtain the full result from the scaling rule. For notational convenience, we will continue to write z as opposed to $x+i$. Let $\hat{B}_t$ be a Brownian excursion in $\mathbb{Q}$. That is, $\hat{B}_t$ is a complex Brownian motion conditioned to stay in the quarter plane for all time $t > 0$. Let Υ denote the infinite corner strip $\Upsilon = \{w \in \mathbb{Q} : \text{Im}(w) < 1\} \cup \{w \in \mathbb{Q} : \text{Re}(w) < x\}$. We begin by calculating the asymptotics as $\epsilon \to 0+$ for the probability, p_1, that $\hat{B}_t$ exits Υ along the ϵ-corner gap $\Gamma_\epsilon = \{w \in \mathbb{Q} : \text{Im}(w) = 1, \text{Re}(w) \in [x, x+\epsilon)\} \cup \{z \in \mathbb{Q} : \text{Re } z = x, \text{Im } z \in [1, 1+\epsilon)\}$. Let τ be the first time $\hat{B}_t$ hits Q_z. We then calculate the asymptotics as $\epsilon \to 0+$ of the probability, p_2, that, starting somewhere in the gap Γ_ϵ, $\hat{B}[0,\infty)$ avoids $J(z,\epsilon)$. And then by the Strong Markov property, the probability $\mathbb{P}^{(\mathbb{Q},0,\infty)}_\alpha\{K \cap J(z,\epsilon) = \emptyset\}$ is asymptotically given by the product of these two events, integrated along the starting positions of the Brownian excursion stopped at the corresponding stopping times.

We begin computing the asymptotics as $\epsilon \to 0+$ of p_1 by finding a conformal transformation sending $\mathbb{H}$ onto Υ. Since Υ is a polygon with four infinite vertices, there exists a Schwartz-Christoffel transformation $f : \mathbb{H} \to \Upsilon$. Let $z_1, \ldots, z_4 \in \mathbb{R}$ be the prevertices of the map f and let $w_1, \ldots, w_4$ be the corresponding vertices with respective angles $\pi\alpha_1, \ldots, \pi\alpha_4$. In our case we have $w_1 = 0$, $w_3 = x + i$ and $w_2 = w_4 = \infty$. Without loss of generality, we may assume $z_4 = \infty$, and then f takes the form

$$f(z) = A + C \int_{z_1}^{z} (\zeta - z_1)^{\alpha_1 - 1} (\zeta - z_2)^{\alpha_2 - 1} (\zeta - z_3)^{\alpha_3 - 1} \ d\zeta. \tag{5.2.10}$$

We can then choose f so that $f(z_1) = 0$, in which case we have $A = 0$. The condition on the α_k is that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2. \tag{5.2.11}$$

It is clear that we must have $\alpha_1 = 1/2$, $\alpha_3 = 3/2$. By choosing an appropriate sequence of increasing polygons Υ_n, such that $\Upsilon_n \to \Upsilon$ as $n \to \infty$ in the Carethadory topology (i.e. choose them such that the angles $\pi\alpha_2^n = \pi\alpha_4^n$ for all n), we can see that we must have $\alpha_2 = \alpha_4 = 0$. Therefore, equation 5.2.10 takes the form

$$f(z) = \int_{z_1}^{z} (\zeta - z_1)^{-1/2} (\zeta - z_2)^{-1} (\zeta - z_3)^{1/2} \ d\zeta. \tag{5.2.12}$$

Let $\tilde{B}_t$ be a Brownian excursion in $\mathbb{H}$ starting at 0 and conditioned to leave $\mathbb{H}$ along the line segment $\Gamma = (z_2, \infty)$. Let $\tilde{\tau} = \inf_{t \geq 0}\{\tilde{B}_t \in \Gamma\}$. Let $H_{\partial\mathbb{H}}(0, x')$, $x' \in \Gamma$ be the boundary Poisson kernel for paths in $\mathbb{H}$, starting at 0 and ending somewhere on the line segment Γ. The probability that $\tilde{B}_t$ ends somewhere along the gap of width 2ϵ, $(z_3 - \epsilon, z_3 + \epsilon)$, is given by

$$\mathbf{P}\{\tilde{B}_{\tilde{\tau}} \in (z_3 - \epsilon, z_3 + \epsilon)\} = c \int_{z_3 - \epsilon}^{z_3 + \epsilon} H_{\partial\mathbb{H}}(0, x) \ dx \tag{5.2.13}$$

$$\sim 2cH_{\partial\mathbb{H}}(0, z_3)\epsilon. \tag{5.2.14}$$

We can use (5.2.13), along with the conformal invariance of Brownian excursion, to calculate the probability p_1. We want to find a power $s_1 \geq 0$ such that $p_1 \asymp \epsilon^{s_1}$. Therefore, for our purposes it is sufficient to calculate the asymptotics as $\epsilon \to 0+$ of the probability that $\hat{B}_\tau$ lands in the slit Γ_ϵ. We proceed by looking at the image of the interval $(z_3 - \epsilon, z_3)$ under the transformation f. To that end, observe that

$$f(z_3 - \epsilon) = C \int_{z_1}^{z_3 - \epsilon} (\zeta - z_1)^{-1/2} (\zeta - z_2)^{-1} (\zeta - z_3)^{1/2} \; d\zeta \tag{5.2.15}$$

$$\sim C(z_3 - z_1)^{-1/2} (z_3 - z_1)^{-1} \int_{z_1}^{z_3 - \epsilon} (\zeta - z_3)^{1/2} \; d\zeta. \tag{5.2.16}$$

Now let γ denote the upper half of semicircle of radius $(z_3 - \epsilon - z_1)/4$ centered at $(z_3 - \epsilon - z_1)/2$, oriented clockwise. An elementary computation then yields

$$\int_{z_1}^{z_3 - \epsilon} (\zeta - z_3)^{1/2} \; d\zeta = \int_\gamma (\zeta - z_3)^{1/2} \; d\zeta \tag{5.2.17}$$

$$= \text{const } \epsilon^{3/2}. \tag{5.2.18}$$

Therefore, using the conformal invariance of Brownian excursion under the map f^{-1}, we can conclude that

$$p_1 = \mathbf{P}\{\hat{B}_\tau \in \Gamma_\epsilon\} \asymp \epsilon^{2/3}. \tag{5.2.19}$$

as $\epsilon \to 0+$. Let $z' \in \Gamma_\epsilon$. We now proceed to calculate the asymptotics for

$$p_2 = \mathbf{P}\{\hat{B}[\tau, \infty) \cap J(x + i, \epsilon) = \emptyset\} \tag{5.2.20}$$

as $\epsilon \to 0+$.

First note that $\hat{B}_t$ can be realized as the stochastic process $X_t + iY_t$, where X_t, Y_t are independent Bessel-3 processes. Therefore, $\mathrm{Re}(\hat{B}_t), \mathrm{Im}(\hat{B}_t) \to \infty$ as $t \to \infty$ a.s. Given $R > 0$, let $Q_R := Q_{Rx+iR}$. Let $B_t = B_t^1 + iB_t^2$ be a complex Brownian motion and let $\tau_R = \inf\{t \geq 0 : B_t \in Q_R\}$. Let $\eta = \inf\{t \geq 0 : B_t \in \partial\mathcal{B}(z,\epsilon)\}$, and let β be the upper rightmost quarter-circle of $\partial\mathcal{B}(z,\epsilon)$. That is, β can be parametrized by $\beta = z + \epsilon e^{it}$, $0 \leq t \leq \pi/2$.

If $z' \in \Gamma_\epsilon$ with $\mathrm{Im}(z') = 1$, then we have

$$\mathbf{P}^{z'}\{\hat{B}[0,\infty) \cap J(z,\epsilon) = \emptyset\} = \lim_{R\to\infty} \mathbf{P}^{z'}\{B[0,\tau_R] \cap J(z,\epsilon) = \emptyset | B[0,\tau_R] \subset \mathbb{Q}\} \quad (5.2.21)$$

$$= \lim_{R\to\infty} \frac{\mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q} \setminus J(z,\epsilon)\}}{\mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q}\}} \quad (5.2.22)$$

Let us consider the denominator in (5.2.22). Starting at z', in order for $B[0,\tau_R] \subset \mathbb{Q}$, either B^1 must reach xR before 0 while $B^2 \geq R$, or vice versa. It is also possible that both B^1 and B^2 reach xR or R, respectively, before hitting 0 without touching Q_R, and then B_t proceeds to touch Q_R without moving a distance $O(1)$ away from Q_R. We can obtain a lower-bound on $\mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q}\}$ by throwing out the curves which do this and then come back and get arbitrarily close to $\partial\mathbb{Q}$ without touching it, and then go back and get close to Q_R. We also throw out all of the curves that do this more than once. Let $\tau_R^1 = \inf\{t \geq 0 : B_t^1 = xR\}$ and let $\tau_R^2 = \inf\{t \geq 0 : B_t^2 = R\}$. In each case we are considering, we must have $\tau_R^1 < \tau_0^1$, $\tau_R^2 < \tau_0^2$, and so Gambler's ruin gives us our lower bound:

$$\mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q}\} \geq c_- \frac{1}{R^2}, \quad (5.2.23)$$

where c_- is a positive constant. In order to obtain our upper bound for this probability, we must take into account all of the events that we threw away to achieve our lower bound. Note that the event $\{B[0,\tau_R] \subset \mathbb{Q}\}$ can be written as $\{\tau_R < \tau_0\}$, and this event can be broken down into the disjoint union of the events $A_1 = \cup_j\{\tau_R^1 < \tau_0^1, \tau_R^2 < \tau_0^2, \tau_R^j = \tau_R\}$ and $A_2 = \cup_j\{\tau_R^1 < \tau_0^1, \tau_R^2 < \tau_0^2, \tau_R^j < \tau_R < \tau_0\}$. The event A_2 consists of curves which pass the line $\mathrm{Re}(w) = xR$ and the line $\mathrm{Im}(w) = R$ and then proceed to touch Q_R before touching $\partial\mathbb{Q}$. But this event consists of a multitude of curves, those which touch Q_R before moving a distance $O(1)$ away from Q_R, and those which move a distance $O(1)$ away from Q_R and then come back to Q_R, and those which move a distance $O(1)$ away from Q_R and come back any multiple of times before touching Q_R. However, the same Gambler's ruin estimate given in (5.2.23) shows that the event that the Brownian motion moves a distance $O(1)$ away from Q_R and then comes back to touch Q_R doesn't contribute asymptotically to the event $\{\tau_R < \tau_0\}$ since there is at most an $O(R^{-3})$ chance that this happens. Also, note that the event that $\tau_R^1 < \tau_0^1$, $\tau_R^2 < \tau_0^2$ and $\tau_0 < \tau_R$ does not contribute asymptotically to the event $\cup_j \{\tau_R^1 < \tau_0^1, \tau_R^2 < \tau_0^2, \tau_R^j = \tau_R\} \cup \{\tau_R^1 < \tau_0^1, \tau_R^2 < \tau_0^2, \tau_R^1 \neq \tau_R^2 \neq \tau_R\}$ by the same Gambler's ruin estimate. Thus, asymptotically, the event $\{\tau_R^1 < \tau_0^1, \tau_R^2 < \tau_0^2\}$ dominates $\mathbf{P}^{z'}\{\tau_R < \tau_0\}$, and our upper bound on the denominator in (5.2.22) is given by

$$\mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q}\} \leq c_+ \frac{1}{R^2},$$

where $c_+ > c_-$ is a positive constant, which might depend on x, and we can conclude

$$c_- \frac{1}{R^2} \leq \mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q}\} \leq c_+ \frac{1}{R^2} \tag{5.2.24}$$

as $R \to \infty$.

The numerator in (5.2.22) is a bit more complicated, but comes down to the same Gambler's ruin estimate as that used to derive the strong approximation of the denominator. As before, let $\eta = \inf\{t \geq 0 : B_t \in \partial\mathcal{B}(z,\epsilon)\}$. Starting at a point on Γ_ϵ, with strictly positive probability, we will have $B_\eta \in \beta$. Starting at a point $w \in \beta$, the Brownian motion can either travel to Q_R before returning back down into the ϵ-corner, or it can go back through the epsilon corner and then come back out any number of times. To achieve our lower bound, we can assume that the Brownian motion travels up and touches Q_R before passing back down through the gap. This is a weak estimate, but it will serve our purposes. Suppose that $w = z + \delta_1 + i\delta_2$, where $0 < \delta_1, \delta_2 < \epsilon$. By an argument similar to that used to find the upper and lower bounds of the denominator in (5.2.22), the Gambler's ruin estimate, along with the strong Markov property applied at η gives us the lower bound

$$\mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q} \setminus J(z,\epsilon)\} \geq C_- \frac{\epsilon^2}{(R-1)^2},$$

where C_- is a positive constant.

To obtain the upper bound, we need to consider the curves which, starting at $w \in \beta$, dip back down below the ϵ-corner and then come back up through to touch Q_R. To begin, notice that starting at a point $z' \in \Gamma_\epsilon$, the probability that the Brownian motion exits $\mathcal{B}(z,\epsilon)$ at a point along β is bounded above by $1/2$. Let $\eta_1 = \eta$, $\tau_1 = \tau$, and for $j = 2,3,\ldots$, let $n_j = \inf\{t \geq \tau_j : B_t \in \partial\mathcal{B}(z,\epsilon)\}$ and $\tau_j = \inf\{t \geq \eta_{j-1} : B_t \in \Gamma_\epsilon\}$. On the event $B_{\eta_j} \in \beta$, the probability that the Brownian motion goes back up to touch Q_R while staying in $\mathbb{Q} \setminus J(z,\epsilon)$ can be broken down into those curves which go up to touch Q_R without coming back down through the ϵ-corner, and those curves which come back down through the ϵ corner and then proceed to go up and hit Q_R. Also, on the event $B_{\eta_j} \in \beta$, it suffices to ignore the event that the Brownian motion dips back down a distance of $O(1)$ in the south-west direction of the corner, since this event doesn't contribute asymptotically. Indeed, if the Brownian motion travels through the ϵ-corner a distance of $O(1)$, it has an $O(\epsilon^{2/3})$ chance up making it back up through the gap, and then an $O(\epsilon^2)$ chance of making it a distance $O(1)$ to the north-east of the ϵ-corner, and this doesn't contribute asymptotically. The reason that the probability of making it a distance of $O(1)$ to the north-east of the Γ_ϵ is $O(\epsilon^2)$ will be discussed below.

Since each time the Brownian motion makes it back down to Γ_ϵ from β, the probability that it makes it back up to β is bounded above by $1/2$, following a similar argument to the one used to obtain the upper bound in the numerator for (5.2.22), we can bound $\mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q} \setminus J(z,\epsilon\}$ from above by

$$\begin{aligned}\mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q} \setminus J(z,\epsilon\} &\leq C \sum_{j=1}^{\infty} 2^{-j} \mathbf{P}^w\{\tau_R^1 < \tau_0^1, \tau_R^2 < \tau_0^2\} \\ &\leq C_+ \frac{\epsilon^2}{(R-1)^2},\end{aligned}$$

where C, C_+ are positive constants. This leads to

$$C_- \frac{\epsilon^2}{(R-1)^2} \leq \mathbf{P}^{z'}\{B[0,\tau_R] \subset \mathbb{Q} \setminus J(z,\epsilon)\} \leq C_+ \frac{\epsilon^2}{(R-1)^2}. \tag{5.2.25}$$

Combining this with (5.2.24), and taking $\limsup_{R\to\infty}$ on the one hand, and $\liminf_{R\to\infty}$ on the other hand, we arrive at

$$\mathbf{P}^{z'}\{\hat{B}[0,\infty) \cap J(z,\epsilon) = \emptyset\} \asymp \epsilon^2,$$

which implies that $p_2 \asymp \epsilon^2$. Therefore, we have shown that

$$\Phi'_{J(z,\epsilon)}(0) \asymp \epsilon^{8/3},$$

and we can conclude that

$$\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}\{K \cap J(z,\epsilon) = \emptyset\} \asymp \epsilon^{8\alpha/3}.$$

□

Remark 5.2.26. We derived the bound $\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}\{K \cap J(z,\epsilon) = \emptyset\} \asymp \epsilon^{8/3}$ in the case $z = x + i$. It is easy to see that for arbitrary z (that is, take $r > 0$ and consider $z = r(x+i)$ for some $x \in \mathbb{R}$), that the scaling rule implies that we can improve the bound slightly to

$$\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}\{K \cap J(z,\epsilon) = \emptyset\} \asymp \left(\frac{\epsilon}{|z|}\right)^{8/3}.$$

And it is easy to show that this gives the desired decay of $\epsilon^{8/3}$ as $\epsilon \to 0+$.

Let us now proceed to prove the following:

Proposition 5.2.27. *Let $z, w \in \mathbb{Q}$ such that $|z| < |w|$ and $J(z,\epsilon) \cap J(w,\epsilon) = \emptyset$. Then*

$$\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}\{K \cap (J(z,\epsilon) \cup J(w,\epsilon)) = \emptyset\} \leq C|z-w|^{-8\alpha/3}\epsilon^{16\alpha/3}.$$

Proof. To prove this Proposition, it suffices to show that

$$\Phi'_{J(z,\epsilon)\cup J(w,\epsilon)}(0) \asymp |w-z|^{-8/3}\epsilon^{16/3}. \tag{5.2.28}$$

Let $\hat{B}_t$ be a Brownian excursion in $\mathbb{Q}$ starting at 0 and ending at ∞. Then the left hand side of (5.2.28) is the probability that $\hat{B}[0,\infty)$ avoids $J(z,\epsilon)$ and $J(w,\epsilon)$. First, the excursion needs to pass through Q_z along the ϵ-corner described in the proof of Proposition 5.2.8. Then starting at a point in the ϵ-corner of z, it must then pass through the ϵ-corner at w and then travel up to infinity. These two events are independent, by the strong Markov property. Note that the probability of passing through the first gap, by Proposition 5.2.8, is $O((\epsilon/|z|)^{8/3})$, and that after temporarily shifting z to 0, and applying the conditioning in the proof of Proposition 5.2.8, the probability of the second event is $O((\epsilon/|w-z|)^{8/3})$. Therefore, we see that

$$\Phi'_{J(z,\epsilon)\cup J(w,\epsilon)}(0) \asymp \frac{\epsilon^{16/3}}{(|z||z-w|)^{8/3}},$$

as $\epsilon \to 0+$, from which it is easy to see that the two point estimate A.2.3 is satisfied. □

Let us now prove the main Theorem of this chapter, that $\dim_H(\mathcal{C}) = 2 - 8/3\alpha$.

Proof of Theorem 5.1.1. Conditions 1 and 3 of Proposition A.2.4 have already been shown (Propositions 5.2.8 and 5.2.27). Now, notice that if $z + i\epsilon/2 \in \mathcal{C}_\epsilon$ and $z + \epsilon/2 \in \mathcal{C}_\epsilon$, then $z \in \mathcal{C}_{2\epsilon}$. Let $\mathcal{B}$ be the open quarter disk of radius $\epsilon/2$ subtending an angle of $\pi/2$ from the line $\mathrm{Re}(w) = \mathrm{Re}(z)$, i.e. $\partial\mathcal{B}$ can be parametrized by $\partial\mathcal{B} = z + \epsilon/2e^{it}$, $0 \leq t \leq \pi/2$. It is easy to see that

$$\mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}\{\mathcal{B} \subset \mathcal{C}_\epsilon\} \sim \mathbb{P}_\alpha^{(\mathbb{Q},0,\infty)}\{z + \epsilon/2, z + i\epsilon/2 \in \mathcal{C}_\epsilon\}$$

as $\epsilon \to 0+$. It is clear that the event on the right-hand side is larger. It can possibly contain hulls K which pass up in between the two vertical or the two horizontal lines of $J(z, 2\epsilon)$ and $J(z + \epsilon/2, \epsilon)$ and $J(z + i\epsilon/2, \epsilon)$. However, the event that this happens is asymptotically insignificant. For example, the probability of the event of passing up through the line segment, l, connecting $z + i3\epsilon/2$ and $z + \epsilon/2 + i\epsilon$ is of $O(\epsilon^{\alpha})$ (see [AC2010]), and then the event of then coming back down (without hitting the vertical lines extending from the points $z + \epsilon/2 + i$ and $z + i3\epsilon/2$) and then passing through the $\epsilon/2$-corner to escape to infinity is at most $O(\epsilon^{8\alpha/3+\alpha})$, leaving an overall chance of at most $O(\epsilon^{14\alpha/3})$ that the hull passes through the line segment l, doesn't touch the vertical lines, returns back through l, and then exits through the $\epsilon/2$-corner. This doesn't contribute asymptotically to the above probability. This, along with the implication that if $z + i\epsilon/2 \in \mathcal{C}_\epsilon$ and $z + \epsilon/2 \in \mathcal{C}_\epsilon$, then $z \in \mathcal{C}_{2\epsilon}$, and Propositions 5.2.8 and 5.2.27, gives the following:

$$\begin{aligned}
\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{\mathcal{B} \subset \mathcal{C}_\epsilon | z \in \mathcal{C}_{2\epsilon}\} &\sim \mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{z + \epsilon/2, z + i\epsilon/2 \in \mathcal{C}_\epsilon | z \in \mathcal{C}_{2\epsilon}\} \\
&\geq \frac{\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{z + i\epsilon/2, z + \epsilon/2 \in \mathcal{C}_\epsilon\}}{\mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\{z \in \mathcal{C}_{2\epsilon}\}} \\
&\geq c\frac{\epsilon^{-8\alpha/3}\epsilon^{16\alpha/3}}{\epsilon^{8\alpha/3}} \\
&= c > 0
\end{aligned}$$

for some positive constant c. This gives condition 2 of Proposition A.2.4, which, along with Propositions 5.2.8 and 5.2.27, is enough to conclude that

$$\dim_H(\mathcal{C}) = \max(2 - 8\alpha/3, 0) \qquad \mathbb{P}_{\alpha}^{(\mathbb{Q},0,\infty)}\text{-a.s.}$$

$\square$

Chapter 6

The fixed irreducible bridge ensemble for self-avoiding walk.

Here we run an i.i.d sequence of n irreducible bridges under the measure on irreducible bridges which gives weight $\beta^{-|\omega|}$ to each irreducible bridge starting at 0. We concatenate the n irreducible bridges and then scale by the reciprocal of the height of the resulting bridge. This gives us a curve in the unit strip $\{z \in \mathbb{H} : 0 < \operatorname{Im}(z) < 1\}$ which starts at 0 and ends at some point along the upper boundary of the strip. If we take $n \to \infty$, this gives us a scaling limit on curves in the unit strip from 0 to a point along the upper boundary. We conjecture a relationship of this scaling limit to that of $SLE_{8/3}$ in the unit strip. We provide a heuristic argument for this conjecture as well as numerical evidence in support of the conjecture. Our evidence allows us to give an estimate for the boundary scaling exponent related to self-avoiding walk.

6.1 Introduction

Consider the set of all infinite upper half-plane SAWs on the lattice $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ rooted at 0, denoted $\mathcal{H}_\infty$, under the distributional limit of the uniform measure on $\mathcal{H}_N$. This measure is defined in Chapter 2, and we denote this measure by $\mathbf{P}_{\mathbb{H},\infty}$. Given $\omega \in \mathcal{H}_\infty$, ω can be decomposed into the concatenation of an i.i.d. sequence of irreducible bridges $\omega^1, \omega^2, \ldots \in \mathcal{I}$. Let $Y_n = Y_n(\omega)$ denote the height of the n-th irreducible bridge in the concatenation, that is $Y_n = \operatorname{Im}\, \omega(|\omega^1 \oplus \cdots \oplus \omega^n|) - \operatorname{Im}(\omega_0)$ for $\omega \in \mathcal{H}_\infty$. We conjecture that there exists $\sigma > 0$ such that,

$$\lim_{n\to\infty} \frac{Y_n}{n^\sigma} = Y, \tag{6.1.1}$$

where the limit is taken in distribution, and Y has the distribution of a stable random variable. Now take $\omega \in \mathcal{H}_\infty$ and let $\hat{\omega} \in \mathcal{I}^n$ denote the concatenation of the first n irreducible bridges in ω. Scale $\hat{\omega}$ by $1/Y_n(\omega)$ and take $n \to \infty$. This gives a scaling limit of curves in the unit strip $S := \{z \in \mathbb{H} : 0 < \mathrm{Im}(z) < 1\}$ starting at 0 and ending anywhere along the upper boundary of the strip. We refer to this as the *fixed irreducible bridge scaling limit*, or *fixed irreducible bridge ensemble*. It is then natural to look for some relationship between the fixed irreducible bridge scaling limit and chordal $SLE_{8/3}$.

The simplest relationship would be the following. Take a curve γ sampled from the fixed irreducible bridge scaling limit. This gives a probability measure on curves in the unit strip. Since these curves can end anywhere along the upper boundary of the unit strip, it is necessary to integrate along the upper boundary of the strip against the conjectured exit density for the scaling limit of SAW in the strip using *SLE* partition functions. Let $\rho(x)$ be the conjectured exit density for the scaling limit of SAW in the strip. Chordal $SLE_{8/3}$ gives a probability measure on curves in the unit strip starting at the origin and ending at some prescribed point along the upper boundary. Thus, it might be reasonable to ask whether γ has distribution given by chordal $SLE_{8/3}$, from 0 to $x + i$, integrated along the density $\rho(x)$. In this paper, we argue that it turns out that the answer to the above question is no, but that this process of scaling the walk to obtain a curve in the unit strip gives chordal $SLE_{8/3}$ integrated over $\rho(x)$ if we weight each of the scaled walks $\hat{\omega}/Y_n(\omega)$, $\hat{\omega} \in \mathcal{I}^n$, by $Y_n(\omega)^p$ before taking the limit $n \to \infty$, where the power p is conjectured to be $-1/\sigma$ for σ defined according to (6.1.1).

6.1.1 Scaling limits and SLE partition functions

In this section we review some conjectured scaling limits of self-avoiding walk, along with *SLE partition functions*, which we will use in what is to come. One, which we have already discussed, is the fixed irreducible bridge ensemble, obtained by considering infinite length self-avoiding walks in the upper half plane starting at 0 and ending at ∞, under the measure $\mathbf{P}_{\mathbb{H},\infty}$ scaling by $1/Y_n(\omega)$, and letting $n \to \infty$.

The next two scaling limits we consider are examples of the *Schramm-Loewner evolution*, introduced by Oded Schramm in [Schramms2000]. Let $D \subset \mathbb{C}$ be a bounded,simply connected domain (other than $\mathbb{C}$) and let $z, w \in \partial D$ be boundary points and $v \in D$ be an interior point. Given $\delta > 0$, let $[z], [w], [v]$ denote the lattice points on $\delta\mathbb{Z}^2$ which are a minimum distance from z, w and v, respectively. One can then consider all SAWs ω in $\delta\mathbb{Z}^2$ beginning at $[z]$ and ending at $[w]$, constrained to stay in D. We weight each walk by $\beta^{-|\omega|}$. The total weight of all such walks is then

$$Z_\delta(D, z; w) = \sum_{\omega \subset D: z \to w} \beta^{-|\omega|}. \tag{6.1.2}$$

We then define a probability measure on all such walks ω in D from $[z]$ to $[w]$ by assigning probability $\beta^{-|\omega|}/Z_\delta(D, z; w)$ to each such walk. The scaling limit as $\delta \to 0+$ is believed to exist and be equal to chordal $SLE_{8/3}$ in D from z to w. We will denote the chordal $SLE_{8/3}$ measure supported on curves $\gamma : [0, t_\gamma] \to \overline{D}$ such that $\gamma(0, t_\gamma) \subset D$, $\gamma(0) = z$, $\gamma(t_\gamma) = w$ by $\mathbf{P}^{chordal}_{D,z,w}$. Of particular interest to us will be the chordal $SLE_{8/3}$ defined as above where D is the unit strip $S := \{z \in \mathbb{H} : 0 < \text{Im } z < 1\}$, $z = 0$, and $w = x + i$, where $x \in \mathbb{R}$. We will denote this probability measure by $\mathbf{P}^{chordal}_{S,0,x+i}$.

One can also consider self-avoiding walks starting at a boundary point $[z]$ and ending at an interior point $[v]$. The resulting scaling limit is thought to be *radial* $SLE_{8/3}$. However, we will not be concerned with radial $SLE_{8/3}$ in this paper.

In the case that $D = \mathbb{H}$, $z = 0$, and $w = \infty$, in order to obtain the scaling limit, one must first find a way to define infinite length SAWs in $\mathbb{H}$. This was done in [LSW2002] and is how the measure $\mathbf{P}_{\mathbb{H},\infty}$ was originally defined. The scaling limit of $\mathbf{P}_{\mathbb{H},\infty}$ as $\delta \to 0+$ is then conjectured to be $\mathbf{P}^{chordal}_{\mathbb{H},0,\infty}$. It is worth mentioning, however, that one can also obtain the probability measure $\mathbf{P}_{\mathbb{H},\infty}$ by a method that is similar in spirit to the method for obtaining the scaling limit for SAW in bounded domains. If we consider the set of all finite length SAWs in $\mathbb{H}$ starting at 0 and weight each such walk ω by $\beta^{-|\omega|}$, then the total weight of all such walks is infinite. If, instead, we weight each such ω by $x^{-|\omega|}$ for $x > \beta$, then the total weight is finite. The limit as $x \to \beta+$ has been shown to exist and to give the same measure on infinite half-plane SAWs as the weak limit on the uniform measures [DGKLP2011].

Finally, let us consider how the normalization factor (6.1.2) depends on the boundary points $z, w \in \partial D$. It is conjectured that there exists a boundary scaling exponent $b > 0$ and a function $H(\partial D, z, w)$ such that as $\delta \to 0+$,

$$Z_\delta(D, z, w) \sim \delta^{2b} H(\partial D, z, w), \tag{6.1.3}$$

and $H(\partial D, z, w)$ is thought to satisfy the following form of conformal covariance. If Φ is a conformal transformation from D onto D', with $\Phi(z) = z'$, $\Phi(w) = w'$, then

$$H(\partial D, z, w) = |\Phi'(z)|^b |\Phi'(z)|^b H(\partial D, z', w'). \tag{6.1.4}$$

See [LSW2002, Lawler2009, Lawler]. Note that in [LSW2002], the boundary scaling exponent is denoted by a, whereas we are denoting it by b.

Recently, it has been shown that there are lattice effects which should persist in the scaling limit for general domains D (see [KL2011]). Therefore, one cannot expect equations (6.1.3) and (6.1.4) to provide a full description of the scaling limit for general domains $D \subset \mathbb{C}$. However, we will be restricting our attention to curves in the domains $\mathbb{H}$ and S, for which there are no lattice effects expected to persist in the scaling limit.

In section 6.3 we will use equation (6.1.4) to derive the predicted exit density for the scaling limit of self-avoiding walks in the unit strip beginning at the origin and ending anywhere along the upper boundary. We will denote the density by $\rho(x)$, where we are assuming that each walk exits the strip at some point $x + i$ with $x \in \mathbb{R}$.

In section 6.2 we state our conjecture about how to obtain chordal $SLE_{8/3}$ from the fixed irreducible bridge ensemble precisely and provide a heuristic argument. The conjecture involves the stability parameter, σ, defined according to (6.1.1). In order to test this conjecture (section 6.4), we require a definite value for σ. We conjecture that $\sigma = 4/3$. In the appendix, we give a heuristic argument in support of this.

6.2 The conjecture

6.2.1 Statement of the conjecture

In order to precisely state our conjecture, we first recall some notations introduced in section 6.1. $\mathbf{P}_{\mathbb{H},N}$ denotes the probability measure on N-step upper-half plane SAWs beginning at 0 defined on the lattice $\mathbb{Z}^2$, and $\mathbf{P}_{\mathbb{H},\infty}$ denotes the probability measure on infinite length SAWs in the upper half plane beginning at 0 and ending at ∞, defined on the lattice $\mathbb{Z}^2$. $\mathbf{P}^{chordal}_{S,0,x+i}$ denotes chordal $SLE_{8/3}$ measure in the unit strip $S := \{z \in \mathbb{H} : 0 < \mathrm{Im}(z) < 1\}$ on curves beginning at 0 and ending at $x + i$, and $\rho(x)$ denotes the conjectured exit density along the upper boundary $\mathrm{Im}(z) = 1$ of the scaling limit for SAW in the unit strip S, starting at 0 and ending anywhere along the upper boundary, which can be derived using SLE partion functions (see Section 6.3.1). Then the conjecture can be stated as follows:

Conjecture 6.2.1. *The fixed irreducible bridge scaling limit of the SAW and chordal $SLE_{8/3}$ in the unit strip S are related by*

$$\lim_{n\to\infty} \frac{\mathbf{E}_{\mathbb{H},\infty}\left[Y_n(\omega)^{-1/\sigma} 1\left(\hat{\omega}/Y_n(\omega) \in E\right)\right]}{\mathbf{E}_{\mathbb{H},\infty}\left[Y_n(\omega)^{-1/\sigma}\right]} = \int_{-\infty}^{\infty} \mathrm{d}x \rho(x) \mathbf{P}^{chordal}_{S,0,x+i}(E), \tag{6.2.2}$$

where E is an event of simple curves in the strip beginning at 0 and ending anywhere along the upper boundary of the strip, $\omega \in \mathcal{H}_\infty$, and $\hat{\omega}$ is ω considered up to its n-th bridge height $Y_n(\omega)$.

So, in practice, we can generate chordal $SLE_{8/3}$ in the unit strip by generating N-step SAWs ω in the half-plane for very large values of N, considered up to height $Y_n(\omega)$ for large n, scaling them to get a curve in the unit strip from 0 to the upper boundary of the strip, and then weighting them by $Y_n(\omega)^{-1/\sigma}$. The conjectured value of σ is 4/3 (See Appendix A.2)

6.2.2 The derivation

In order to derive Conjecture 6.2.1, we fix two heights y_1 and y_2, $y_1 < y_2$, which we think of as order 1, and a large real number $L > 0$. We will then only consider curves which have a bridge point in the region $A = \{z \in \mathbb{H} : y_1 L \leq \text{Im}(z) \leq y_2 L\}$. Let $\mathcal{I}^n = \mathcal{I} \times \cdots \times \mathcal{I}$ (n times) be the set of all $\omega \in \mathcal{H}_\infty$ such that $\omega = \omega^1 \oplus \cdots \oplus \omega^n$, with $\omega^1, \ldots, \omega^n \in \mathcal{I}$, i.e. the set of all concatenations of n irreducible bridges beginning at the origin. Recall that if $\hat{\omega} \in \mathcal{I}^n$ and $\mathcal{H}_\infty(\hat{\omega})$ denotes the set of all $\omega \in \mathcal{H}_\infty$ such that $\omega = \hat{\omega} \oplus \tilde{\omega}$ with $\tilde{\omega} \in \mathcal{H}_\infty$, then we have

$$\mathbf{P}_{\mathbb{H},\infty}\left(\mathcal{H}_\infty(\hat{\omega})\right) = \beta^{-|\hat{\omega}|}.$$

Therefore, the total weight of all SAWs in $\mathcal{H}_\infty$ with a bridge pont in A is

$$Z(A) = \sum_{n=0}^{\infty} \sum_{\hat{\omega} \in \mathcal{I}^n} \beta^{-|\hat{\omega}|} 1\left(Y_n(\omega) \in [y_1 L, y_2 L]\right). \tag{6.2.3}$$

Now let E be an event of simple curves in the unit strip S starting at 0 and ending anywhere along the upper boundary of the strip. We define the probability of the event E to be $N(E, A)/Z(A)$, where

$$N(E, A) = \sum_{n=0}^{\infty} \sum_{\hat{\omega} \in \mathcal{I}^n} \beta^{-|\hat{\omega}|} 1\left(Y_n(\omega) \in [y_1 L, y_2 L]\right) 1\left(\hat{\omega}/Y_n(\omega) \in E\right). \tag{6.2.4}$$

According to the definition of $\mathbf{P}_{\mathbb{H},\infty}$, we have

$$N(E, A) = \sum_{n=0}^{\infty} \mathbf{E}_{\mathbb{H},\infty}\left[1\left(Y_n(\omega) \in [y_1 L, y_2 L]\right) 1\left(\hat{\omega}/Y_n(\omega) \in E\right)\right]. \tag{6.2.5}$$

. Since we have fixed L to be a very large number, this forces each term in the above sum to be zero other than those corresponding to very large values of n. Then, according to 6.1.1, if we fix $N \in \mathbb{N}$ large enough, $N^{-\sigma} Y_N$ should have approximately the same distribution as $n^{-\sigma} Y_n$ for all n sufficiently large. Therefore, the condition $Y_n(\omega) \in [y_1 L, y_2 L]$ can be replaced with the condition (for very large fixed N)

$$y_1 L n^{-\sigma} N^{\sigma} \leq Y_N(\omega) \leq y_2 L n^{-\sigma} N^{\sigma}. \tag{6.2.6}$$

Furthermore, since for large values of n, the distribution of $\hat{\omega}/Y_n(\omega)$ approaches the distribution of a curve pulled from the fixed irreducible bridge ensemble, the condition $\hat{\omega}/Y_n(\omega) \in E$ can be replaced with the condition $\hat{\omega}/Y_N(\omega) \in E$. This, along with (6.2.5) and (6.2.6) lead to

$$N(E,A) \approx \sum_{n=0}^{\infty} \mathbf{E}_{\mathbb{H},\infty}\left[1\left(\left(N^{\sigma}\frac{y_1 L}{Y_N(\omega)}\right)^{1/\sigma} \leq n \leq \left(N^{\sigma}\frac{y_2 L}{Y_N(\omega)}\right)^{1\sigma}\right) 1\left(\hat{\omega}/Y_N(\omega) \in E\right)\right]. \tag{6.2.7}$$

Now we move the sum on n inside the expectation and consider

$$\sum_{n=0}^{\infty} 1\left(\left(N^{\sigma}\frac{y_1 L}{Y_N(\omega)}\right)^{1/\sigma} \leq n \leq \left(N^{\sigma}\frac{y_2 L}{Y_N(\omega)}\right)^{1/\sigma}\right).$$

This sum is easily approximated.

$$\begin{aligned}\sum_{n=0}^{\infty} 1\left(\left(N^{\sigma}\frac{y_1 L}{Y_N(\omega)}\right)^{1/\sigma} \leq n \leq \left(N^{\sigma}\frac{y_2 L}{Y_N(\omega)}\right)^{1/\sigma}\right) &\approx N L^{1/\sigma} Y_N(\omega)^{-1/\sigma}(y_2^{1/\sigma} - y_1^{1/\sigma}) \\ &= c N L^{1/\sigma} Y_N(\omega)^{-1/\sigma},\end{aligned}$$

where $c = y_2^{1/\sigma} - y_1^{1/\sigma}$. The factor of $cNL^{1/\sigma}$ will cancel out of both numerator and denominator, and then taking the limit of $N(E,A)/Z(A)$ as $N \to \infty$, we are left with

$$\lim_{N\to\infty} \frac{N(E,A)}{Z(A)} = \lim_{n\to\infty} \frac{\mathbf{E}_{\mathbb{H},\infty}\left[Y_n(\omega)^{-1/\sigma} 1\left(\hat{\omega}/Y_n(\omega) \in E\right)\right]}{\mathbf{E}_{\mathbb{H},\infty}\left[Y_n(\omega)^{-1/\sigma}\right]}. \tag{6.2.8}$$

Next, we decompose $N(E,A)$ by the value of the bridge heights. Given a SAW $\omega \in \mathcal{H}_{\infty}$, let $D = D(\omega)$ be the set of bridge heights. That is, $D(\omega)$ is the set of all integers $y \geq 0$ such that there exists $n = 0, 1, \ldots$ such that $Y_n(\omega) = y$. Then we have

$$\begin{aligned}
N(E, A) &= \sum_{n=0}^{\infty} \sum_{\hat{\omega} \in \mathcal{I}^n} \beta^{-|\hat{\omega}|} 1\left(\hat{\omega}/Y_n(\omega) \in E\right) 1\left(Y_n(\omega) \in [y_1 L, y_2 L]\right) \\
&= \sum_{y \in \mathbb{Z} \cap [y_1 L, y_2 L]} \sum_{n=0}^{\infty} \sum_{\hat{\omega} \in \mathcal{I}^n} \beta^{-|\hat{\omega}|} 1(\hat{\omega}/y \in E) 1(Y_n = y) \\
&= \sum_{y \in \mathbb{Z} \cap [y_1 L, y_2 L]} \mathbf{P}_{\mathbb{H},\infty}(\hat{\omega}/y \in E, y \in D) \\
&= \sum_{y \in \mathbb{Z} \cap [y_1 L, y_2 L]} \mathbf{P}_{\mathbb{H},\infty}(\hat{\omega}/y \in E | y \in D) \mathbf{P}_{\mathbb{H},\infty}(y \in D).
\end{aligned}$$

Similarly, we find that

$$Z(A) = \sum_{y \in \mathbb{Z} \cap [y_1 L, y_2 L]} \mathbf{P}_{\mathbb{H},\infty}(y \in D).$$

In [DGKLP2011], it was shown that conditioning on the event that a SAW $\omega \in \mathcal{H}_\infty$ has a bridge height at y and considering the walk up to height y gives the law for self-avoiding walk in the strip $\{z \in \mathbb{H} : 0 < \text{Im } z < y\}$. Therefore, by taking y large enough, and scaling the walk by $1/y$, one should expect to get a distribution approaching that of SAW in the unit strip S starting at 0 and ending anywhere along the upper boundary of the strip, in the scaling limit. It follows that if we sum $\mathbf{P}_{\mathbb{H},\infty}(\hat{\omega}/y \in E | y \in D)$ over all $y \in \mathbb{Z} \cap [y_1 L, y_2 L]$, then take the limit $L \to \infty$, which is effectively the same as taking the limit $N \to \infty$ in 6.2.8, $N(E, A)/Z(A)$ should converge to the law for $SLE_{8/3}$ in the unit strip, starting at 0 and ending at $x + i$, $x \in \mathbb{R}$, integrated over the density $\rho(x)$. In other words, we have

$$\begin{aligned}
\lim_{L \to \infty} \frac{N(E, A)}{Z(A)} &= \lim_{L \to \infty} \frac{\sum_{y \in \mathbb{Z} \cap [y_1 L, y_2 L]} \mathbf{P}_{\mathbb{H},\infty}(\hat{\omega}/y \in E | y \in D) \mathbf{P}_{\mathbb{H},\infty}(y \in D)}{\sum_{y \in \mathbb{Z} \cap [y_1 L, y_2 L]} \mathbf{P}_{\mathbb{H},\infty}(y \in D)} && (6.2.9) \\
&\approx \lim_{L \to \infty} \frac{c \mathbf{P}_{\mathbb{H},\infty}(\hat{\omega}/L \in E | L \in D) \mathbf{P}_{\mathbb{H},\infty}(L \in D)}{c \mathbf{P}_{\mathbb{H},\infty}(L \in D)} && (6.2.10) \\
&= \int_{-\infty}^{\infty} \mathrm{d}x \rho(x) \mathbf{P}^{chordal}_{S,0,x+i}(E). && (6.2.11)
\end{aligned}$$

This completes the derivation of conjecture 6.2.1.

6.3 SLE predictions of random variables

Here we will derive a conjecture for the exit density $\rho(x)$, and use this conjecture, along with $SLE_{8/3}$ theory in order to derive a conjecture for the cumulative distribution function for the right most excursion of SAW in the unit strip $S := \{z \in \mathbb{H} : 0 < \mathrm{Im}(z) < 1\}$, beginning at 0 and ending anywhere along the upper boundary, in the scaling limit. The calculations found in this section were carried out in the paper [DGKLP2011], but we include them for the purpose of self-containment.

6.3.1 The density function $\rho(x)$

In this section we will use *SLE* partition functions to derive a conjecture for the exit density, $\rho(x)$, for the scaling limit of self-avoiding walk defined on the unit strip, along the upper boundary. Recall that for a simply connected domain D and points $z, w \in \partial D$, the *SLE* partition function $H(D, z, w)$ satisfies the conformal covariance property (6.1.4). It was predicted in [LSW2002] that the boundary scaling exponent for SAW is $b = 5/8$. Using this value, the conformal covariance property takes the following form: If Φ is any conformal transformation, then

$$H(D, z, w) = |\Phi'(z)\Phi'(w)|^{5/8} H(\Phi(D), \Phi(z), \Phi(w)). \tag{6.3.1}$$

This defines $H(D, z, w)$ up to specifying it for a particular choice of domain D and boundary points z and w. The convention we follow is of taking $H(\mathbb{H}, 0, 1) = 1$.

First note that if we take Φ to be a dilation $\Phi(z) = xz$, for $x \in \mathbb{R} \setminus \{0\}$, then by (6.3.1), we have

$$H(\mathbb{H}, 0, x) = \left(\frac{1}{x^2}\right)^{5/8} = \frac{1}{x^{5/4}}. \tag{6.3.2}$$

Therefore we can calculate $H(S, 0, x+i)$ by considering the conformal map $f : S \to \mathbb{H}$ such that $f(0) = 0$, $f(x+i) = -e^{\pi x} - 1$, given by $f(z) = e^{\pi z} - 1$. We have $|f'(0)| = \pi$

and $|f'(x+i)| = \pi e^{\pi x}$. Thus,

$$H(S,0,x+i) = |f'(0)|^{5/8} |f'(x+i)|^{5/8} H(\mathbb{H},0,-e^{\pi x}-1)$$
$$= \left[\frac{\pi^2 e^{\pi x}}{(1+e^{\pi x})^2}\right]^{5/8} = \left[\frac{\pi^2}{\cosh^2(\pi x/2)}\right]^{5/8}.$$

According to (6.1.3), this shows that the probability density function $\rho(x)$ should be given by

$$\rho(x) = c\left[\cosh\left(\frac{\pi x}{2}\right)\right]^{-5/4}, \tag{6.3.3}$$

where c is a normalization constant.

6.3.2 The right-most excursion

Given a SAW ω defined in the unit strip S with lattice spacing δ, let $X(\omega) = \max_j \mathrm{Re}\ \omega(j)$ denote the *rightmost excursion* of ω, i.e. the right-most point on the SAW in the strip. Based on the results of Section 6.3.1, we conjecture that, in the scaling limit, X has distribution given by

$$\lim_{\delta\to 0+} \mathbf{P}\left(X < \xi\right) = \int_{-\infty}^{\infty} \mathbf{P}^{chordal}_{S,0,x+i}\left(\max_t \mathrm{Re}\ \gamma(t) < \xi\right)\rho(x)\,\mathrm{d}x, \tag{6.3.4}$$

where $\gamma(t)$ is an $SLE_{8/3}$ curve in the unit strip, starting at 0 and ending at $x+i$, and $\rho(x)$ is given by (6.3.3). To calculate $\mathbf{P}^{chordal}_{S,0,x+i}(\max_t \mathrm{Re}(\gamma(t)) < \xi)$, we use the following form of *conformal invariance*: If D is a simply connected domain, $z, w \in \partial D$, and $f : D \to D'$ is a conformal transformation,

$$\mathbf{P}^{chordal}_{D,z,w}\left(\gamma[0,\infty)\cap A = \emptyset\right) = \mathbf{P}^{chordal}_{D',f(z),f(w)}\left(\tilde{\gamma}[0,\infty)\cap f(A) = \emptyset\right), \tag{6.3.5}$$

where $\mathbf{P}^{chordal}_{D,z,w}$ denotes chordal $SLE_{8/3}$ measure in D, starting at z and ending at w, $\mathbf{P}^{chordal}_{D',f(z),f(w)}$ denotes chordal $SLE_{8/3}$ measure in D', starting at $f(z)$ and ending at $f(w)$, and A is a closed set such that $A \subset \overline{D}$, $z, w \notin A$, $A \cap \overline{D} \subset \partial D$ and $D \setminus A$ is simply connected.

Conformal invariance is built into the definition of every SLE_κ measure. However, $SLE_{8/3}$ measure also satisfies the following restriction property: If D is a simply connected domain, $z, w \in \partial D$, and $D' \subset D$ is another simply connected domain with $z, w \in \partial D'$, then

$$\mathbf{P}^{chordal}_{D,z,w}\left(\gamma[0,\infty) \cap A = \emptyset | \gamma[0,\infty) \subset D'\right) = \mathbf{P}^{chordal}_{D',z,w}\left(\tilde{\gamma}[0,\infty) \cap A = \emptyset\right), \tag{6.3.6}$$

where A is as in (6.3.5), along with the assumption that $A \subset \overline{D'}$. Recall that for the family of restriction measures $\mathbb{P}_\alpha$, we have the following restriction formula:

$$\mathbb{P}_\alpha\left(K \cap A = \emptyset\right) = \Phi_A'(0)^\alpha, \tag{6.3.7}$$

for some real number α, where K is a restriction hull and Φ_A is the unique conformal transformation mapping $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with $\Phi_A(0) = 0$, $\Phi_A(\infty) = \infty$, and $\Phi_A(z) = z + o(1)$ as $z \to \infty$.

In the case of $SLE_{8/3}$, it is known that $\alpha = 5/8$, and therefore we have

$$\mathbf{P}^{chordal}_{\mathbb{H},0,\infty}\left(\gamma[0,\infty) \cap A = \emptyset\right) = \Phi_A'(0)^{5/8}. \tag{6.3.8}$$

It is also well known that the map $f(z) = e^{\pi x}$ defines a conformal transformation from the unit strip to the half-plane $\mathbb{H}$ satisfying $f(0) = 1$, $f(x+i) = -e^{\pi x}$. Therefore, the map

$$\Psi_x(z) = \frac{e^{\pi z} - 1}{e^{\pi z} + e^{\pi x}} \tag{6.3.9}$$

defines a conformal transformation from S onto $\mathbb{H}$ with $\Psi_x(0) = 0$ and $\Psi_x(x+i) = \infty$. It follows then from (6.3.5) that if $x < \xi$,

$$\begin{aligned}\mathbf{P}^{chordal}_{S,0,x+i}\left(\max_t \mathrm{Re}(\gamma(t)) < \xi\right) &= \mathbf{P}^{chordal}_{S,0,x+i}\left(\gamma[0,\infty) \cap \{z \in S : \mathrm{Re}(z) \geq \xi\} = \emptyset\right) \\ &= \mathbf{P}^{chordal}_{\mathbb{H},0,\infty}\left(\tilde{\gamma}[0,\infty) \cap \Psi_x\left(\{z \in S : \mathrm{Re}(z) \geq \xi\}\right) = \emptyset\right).\end{aligned}$$

Let $A = \Psi_x\left(\{z \in S : \mathrm{Re}(z) \geq \xi\}\right)$. Then we can write $A = \{z \in \mathbb{H} : |z - c(x,\xi)| \leq$

$a(x,\xi)\}$, where

$$c(x,\xi) = \frac{1}{2}\left(\frac{e^{\pi\xi}+1}{e^{\pi\xi}-e^{\pi x}} + \frac{e^{\pi\xi}-1}{e^{\pi\xi}+e^{\pi x}}\right)$$
$$a(x,\xi) = \frac{1}{2}\left(\frac{e^{\pi\xi}+1}{e^{\pi\xi}-e^{\pi x}} - \frac{e^{\pi\xi}-1}{e^{\pi\xi}+e^{\pi x}}\right).$$

In this case we can write down Φ_A explicitly. We have

$$\Phi_A(z) = (z - c(x,\xi)) + \frac{a(x,\xi)^2}{z - c(x,\xi)}. \tag{6.3.10}$$

Evaluating the derivative of (6.3.10) at 0 and using (6.3.8), we find that

$$\begin{aligned}\mathbf{P}^{chordal}_{S,0,x+i}\left(\max_t \operatorname{Re}(\gamma(t)) < \xi\right) &= \Phi_A'(0)^{5/8} \\ &= \left[1 - \left(\frac{a(x,\xi)}{c(x,\xi)}\right)^2\right]^{5/8}.\end{aligned}$$

Therefore, we can calculate the distribution of the right most excursion of SAW in the strip in the scaling limit by

$$\lim_{\delta\to 0+} \mathbf{P}\left(X < \xi\right) = \int_{-\infty}^{\xi} \left[1 - \left(\frac{a(x,\xi)}{c(x,\xi)}\right)^2\right]^{5/8} \rho(x)\,\mathrm{d}x,$$

where $\rho(x)$ is given by (6.3.3). Thus, by our Conjecture, 6.2.1, we should have

$$\lim_{n\to\infty} \frac{\mathbf{E}_{\mathbb{H},\infty}\left[Y_n(\omega)^{-1/\sigma} 1(\max_j \omega_j / Y_n(\omega) < \xi)\right]}{\mathbf{E}_{\mathbb{H},\infty}\left[Y_n(\omega)^{-1/\sigma}\right]} = \int_{-\infty}^{\xi} \left[1 - \left(\frac{a(x,\xi)}{c(x,\xi)}\right)^2\right]^{5/8} \rho(x)\,\mathrm{d}x. \tag{6.3.11}$$

6.4 Simulations

The pivot algorithm provides us with a fast Monte Carlo algorithm for simulating the self-avoiding walk in the full plane or the half-plane. It has also recently been shown in [DGKLP2011] that the pivot algorithm can be used to simulate self-avoiding walks in the strip S. Taking lattice effects into account (see [KL2011]), it should also be possible to simulate the self-avoiding walk in other domains using the pivot algorithm. Recently, Nathan Clisby has developed a very fast implementation of the pivot algorithm, [Clisby2010], and that is the algorithm that we use for our simulations.

We use the pivot algorithm to generate self-avoiding walks in the half-plane with number of steps $N = 1000$K. Each iteration of the algorithm is highly correlated, so there is no point in sampling each iteration. Instead, we sample every 100 iterations. In this way, we generated $144,000$K samples.

We first test the conjectured density $\rho(x)$ given by (6.3.3) against the sampled data. We take $n = 100$, sample self-avoiding walks in the half-plane, considering them up to their 100th bridge point, and then scale them by $1/Y_n$ to get a curve in the unit strip. To test the exit density of these curves against $\rho(x)$, we split the interval $[-3, 3]$ into 600 equal parts of length $\mathrm{d}x = 0.01$. We then plot a histogram (see Figure 6.1) by summing the weights $Y_n^{-1/\sigma}$ for each curve γ sampled which satisfies $x \le \mathrm{Re}(\omega(s)/Y_n) < x + \mathrm{d}x$, divided by the sum of the weights $Y_n^{-1/\sigma}$ for every curve sampled. Here we are using s to denote the time at which ω reaches height Y_n. We have also plotted a histogram of the exit density of the curves in the strip obtained from our samples by normalizing by the number of samples generated instead of the sum of the weights $Y_n^{-1/\sigma}$ in order to show that we do not get the conjectured exit density ρ.

Next we test the conjecture by making a prediction for the scaling exponent b in (6.1.3) and (6.1.4). We do this by plotting the log of $\mathbf{E}_N[Y_n^{-1/\sigma} 1(x \le \mathrm{Re}(\omega(s)/Y_n) < x + \mathrm{d}x)]$ versus the log of $\cosh^{-2}(\pi(x + \mathrm{d}x/2)/2)$. We take evenly spaced values of the interval $[-1.90, 1.90]$ with spacing $\mathrm{d}x = 0.01$. By Conjecture 6.2.1, we should have

$$\log\left(\mathbf{E}_N[Y_n^{-1/\sigma} 1(x \le \mathrm{Re}(\omega(s)/Y_n) < x + \mathrm{d}x)]\right) = b \log\left(\cosh^{-2}(\pi(x + \mathrm{d}x/2)/2)\right) + \mathrm{const}. \tag{6.4.1}$$

Therefore, the data points should lie on a line. The slope of the line should be b,

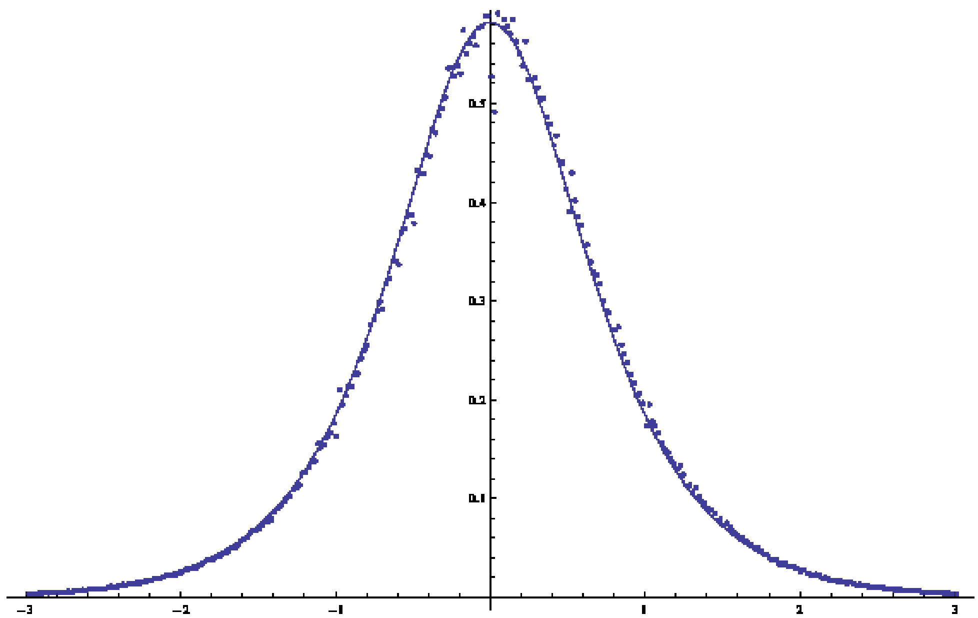

Figure 6.1. Histogram of exit points with the appropriate weighting along the upper boundary of the strip for the fixed irreducible bridge ensemble. The conjectured density $\rho(x)$ is represented by the solid curve, while the histogram is represented by the data points.

which is conjectured to be $5/8 = 0.625$.

The line shown in Figure 6.3 is a least-squares fit to the data, the slope of which is 0.625303. If we let b denote the slope of our least-squares fit, then comparing b with the conjectured value, we have

$$b - 5/8 = 0.000303. \tag{6.4.2}$$

We have also plotted a log-log graph of the expected value of $1(x \leq \mathrm{Re}(\omega(s)/Y_n) < x + \mathrm{d}x)$ versus $\cosh^{-2}(\pi(x + \mathrm{d}x/2)/2$ by calculating the expected value through the number of samples as opposed to summing the weights $Y_n^{-1/\sigma}$. This should be compared to Figure 6.3. The slope of the least-squares fit in this case is 0.444367. We don't expect this to mean anything, since if the log-log plot of the sample points were linear, it would imply that the exit density of the fixed irreducible bridge ensemble is harmonic measure raised to some power, and we do not expect this to be true. In

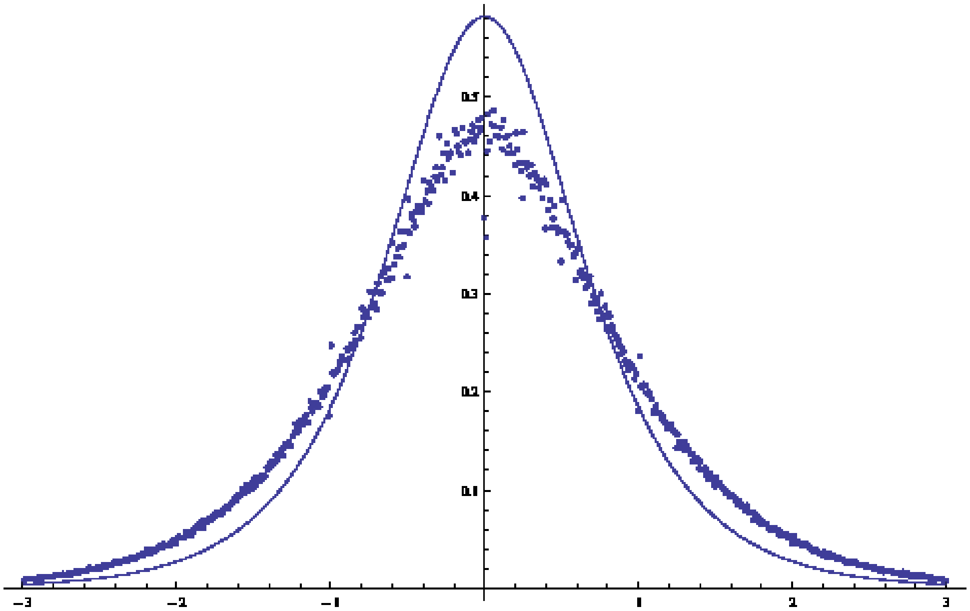

FIGURE 6.2. Histogram of exit points along the upper boundary of the strip obtained by normalizing by the number of samples generated, as opposed to normalizing by the sum of the weights $Y_n^{-1/\sigma}$. Once again $\rho(x)$ is represented by the solid curve.

fact, a careful observation of Figure 6.4

Next, we perform numerical tests on the rightmost excursion, which we are denoting by X. After generating our self-avoiding walks in the half-plane with $N = 1000K$ steps, and considering them up to their 100th bridge point (i.e. taking $n = 100$), we scale the walks by $1/Y_n$ and weight the probability measure by $Y_n^{-1/\sigma}$. We denote the probability obtained in this manner by $\mathbf{P}_{N,n}$. Of course this depends on the number of steps in the walk, as well as the value of n. But for large enough values of N, n, this measure should look very close to the fixed irreducible bridge measure. Conjecture 6.2.1 then states that $\lim_{N,n\to\infty} \mathbf{P}_{N,n} = \mathbf{P}^{chordal}_{S,0,x+i}$, integrated against $\rho(x)$. Given $\xi \geq 0$, by equation (6.3.11), we should have

$$\mathbf{P}_{N,n}(X < \xi) \approx \int_{-\infty}^{\xi} \left[1 - \left(\frac{a(x,\xi)}{c(x,\xi)}\right)^2\right]^{5/8} \rho(x)\, \mathrm{d}x. \tag{6.4.3}$$

We use numerical integration to calculate the right hand side of (6.4.3). Figure 6.5

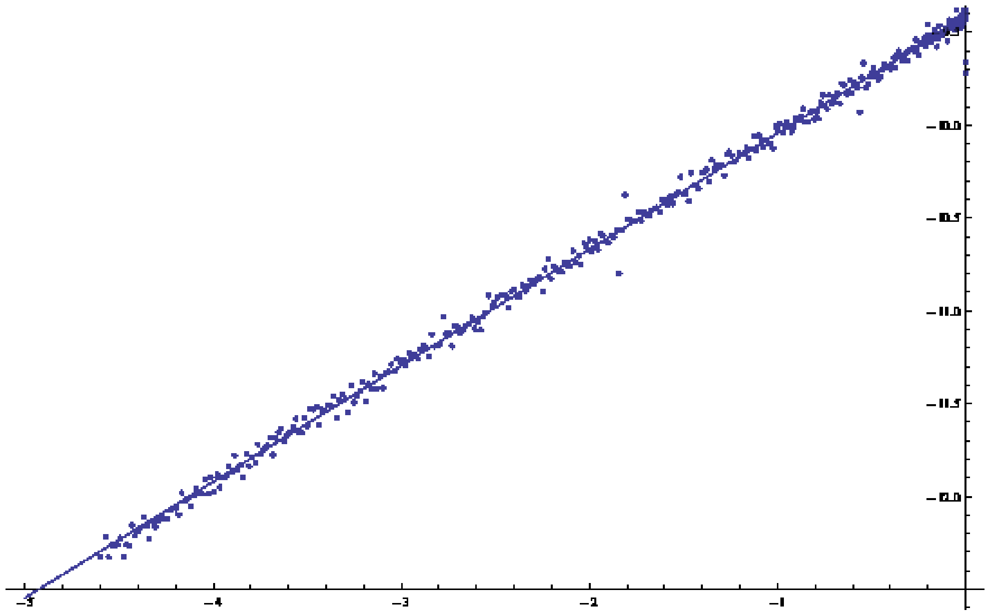

FIGURE 6.3. log-log plot of $\mathbf{E}_N[Y_n^{-1/\sigma} 1(x \leq \mathrm{Re}(\omega(s)/Y_n) < x + \mathrm{d}x)]$ versus $\cosh^{-2}(\pi(x + \mathrm{d}x/2)/2)$. The slope of the least-squares fit is 0.625303.

shows a plot of the cumulative distribution function for X under the measure $\mathbf{P}_{N,n}$ obtained from our simulations, along with the conjectured cumulative distribution function for X given by the right hand side of (6.4.3), for values of ξ between 0 and 5. In the scale of the figure, the two curves look almost identical. In Figure 6.6, we plot the difference between the simulated cdf for X and the conjectured cdf for X.

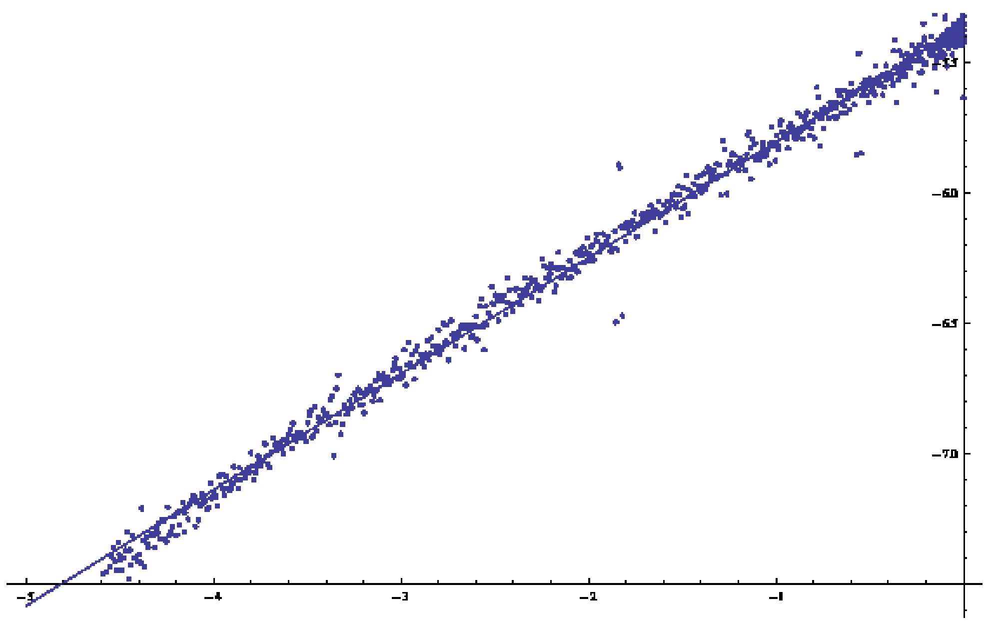

FIGURE 6.4. log-log plot of $\mathbf{E}_N[1(x \leq \mathrm{Re}(\omega(s)/Y_n) < x + \mathrm{d}x)]$ versus $\cosh^{-2}(\pi(x + \mathrm{d}x/2)/2)$. Without taking the weights $Y_n^{-1/\sigma}$ into account, the slope of the least-squares fit is 0.444367.

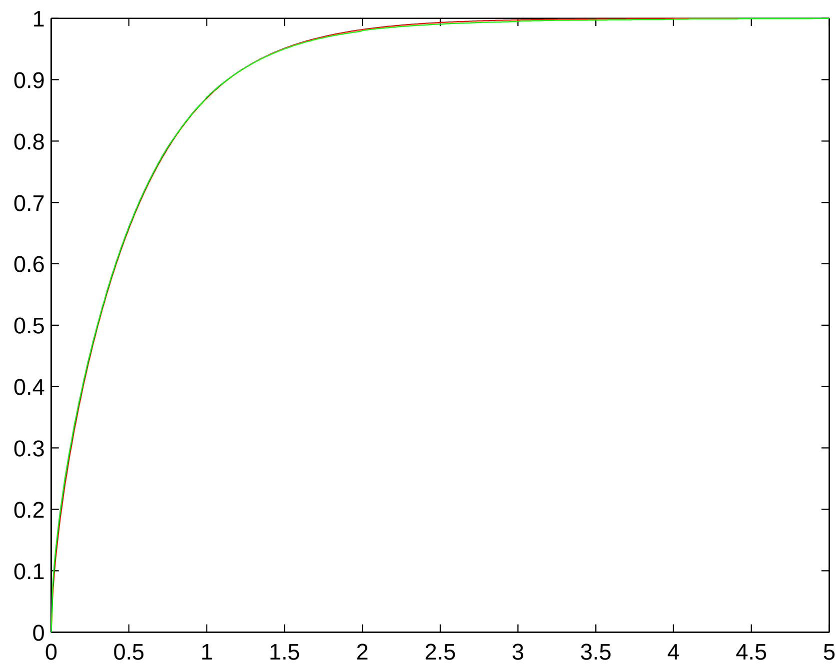

FIGURE 6.5. Plot of the conjectured cdf for the rightmost excursion of SAW in the strip in the scaling limit as $\delta \to 0+$ and the simulated rightmost excursion for SAW in the fixed irreducible bridge ensemble. The conjectured cdf is colored in red, while the simulated cdf is colored in green. In the scale of the image, it is difficult to see the difference.

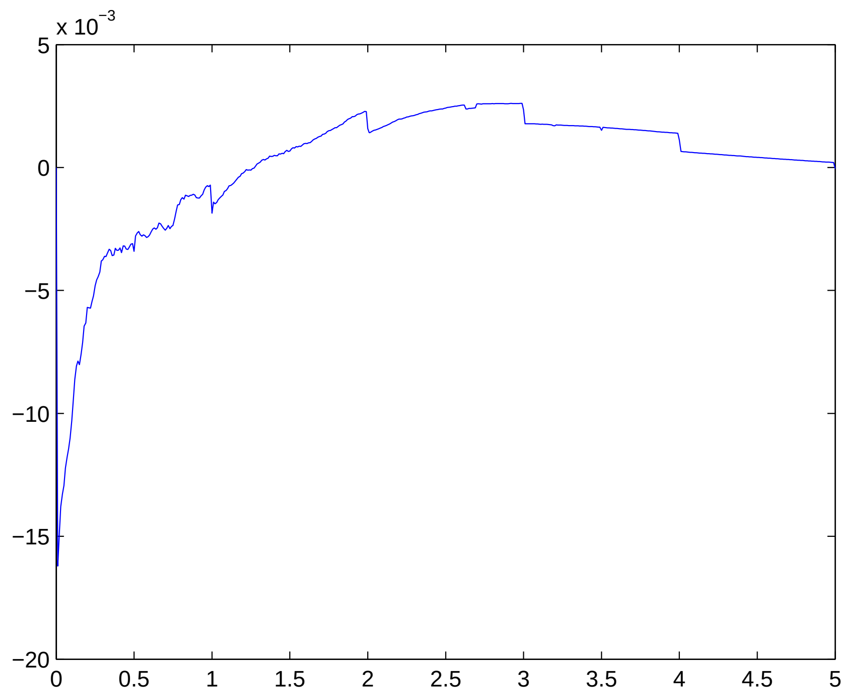

FIGURE 6.6. Plot of the difference in values for the conjectured cdf for the rightmost excursion and the simulated cdf for the rightmost excursion.

Appendix A

All the material that could not make it into the main text

A.1 A subadditivity result

We give here a self-contained proof of a standard result about subadditivity which we use frequently throughout the main text, especially in section 2.1. This result can also be found in [MS1993] and [Lawler2008]

Proposition A.1.1. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers which is subadditive, that is, $a_n + a_m \leq a_{n+m}$. Then the limit $\lim_{n\to\infty} a_n$ exists in $[-\infty, \infty)$ and is equal to*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{k\geq 1} \frac{a_k}{k}. \tag{A.1.2}$$

Proof. It suffices to show that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \frac{a_k}{k} \tag{A.1.3}$$

for all $k \geq 1$. For then, taking the $\liminf_{n\to\infty}$ of both sides of (A.1.3) gives the desired result.

Given $k \geq 1$, let $A_k = \max_{1\leq r\leq k} a_r$. For a fixed $n \in \mathbb{N}$ and k with $1 \leq k \leq n$, let j be the largest integer less than or equal to n/k. Then we can write

$$n = jk + r \text{ for } 1 \leq r \leq k.$$

Subadditivity then gives

$$a_n \leq ja_k + a_r \leq \frac{n}{k} a_k + A_k. \tag{A.1.4}$$

Dividing (A.1.4) by n and taking the $\limsup_{n\to\infty}$ then gives the desired result. $\square$

A.2 Hausdorff dimension of random sets in $\mathbb{C}$

Let m denote Lebesgue measure on $\mathbb{C}$. Suppose $(C_\epsilon)_{\epsilon>0}$ is a collection of random subsets of the complex plane under some probability measure $\mathbf{P}$, and that for $\epsilon < \epsilon'$, we have $C_\epsilon \subset C_{\epsilon'}$. Then if $C = \bigcap C_\epsilon$, define the following conditions (where we use the notation $f(\epsilon) \asymp g(\epsilon)$ to mean that there are positive constants c_1 and c_2, independent of epsilon, z, and w, such that $c_1 f(\epsilon) \leq g(\epsilon) \leq c_2 f(\epsilon)$):

(i) For all $z \in \mathbb{C}$,

$$\mathbf{P}\{z \in C_\epsilon\} \asymp \epsilon^s \tag{A.2.1}$$

(ii) There exists $c > 0$ such that for all $z \in \mathbb{C}$,

$$\mathbf{P}\{m(C_\epsilon \cap \mathcal{B}(z, \epsilon)) \geq c\epsilon^2 | z \in C_\epsilon\} \geq c > 0 \tag{A.2.2}$$

(iii) There exists $c > 0$ such that for all $z, w \in \mathbb{C}$,

$$\mathbf{P}\{z, w \in C_\epsilon\} \leq c\epsilon^{2s}|z - w|^{-s} \tag{A.2.3}$$

Proposition A.2.4. *If conditions (i) and (ii) hold, then a.s.* $\dim_H(C_\epsilon) \leq 2 - s$. *If conditions (i) and (iii) hold, then with some strictly positive probability,* $\dim_H(C_\epsilon) \geq 2 - s$.

Proof. It suffices to show that the result holds in $[0, 1]^2 = [0, 1] + i[0, 1]$. For the upper bound, we create a covering of the box $[0, 1]^2$ by open balls of radius ϵ. To that end, let $(\mathcal{B}_i)$ be a collection of $4\epsilon^{-2}$ open balls of radius ϵ which cover $[0, 1]^2$. By condition (i), the probability that the center of ball $\mathcal{B}_i$ is in C_ϵ is bounded above by a constant times ϵ^s. Let z_i denote the center of $\mathcal{B}_i$ Then

$$\mathbf{P}\{C_\epsilon \cap \mathcal{B}_i \neq \emptyset\} \leq C'\epsilon^s \tag{A.2.5}$$

for some constant $C' > 0$. Thus, if we let $N(\epsilon)$ denote the (random) number of balls of radius ϵ required to cover C_ϵ, (A.2.5) then gives us

$$\begin{aligned}\mathbf{E}[N(\epsilon)] &= \sum_{i=1}^{4\epsilon^{-2}} \mathbf{P}\{C_\epsilon \cap \mathcal{B}_i \neq \emptyset\} \\ &\leq C\epsilon^{s-2}\end{aligned}$$

for a positive constant C. Therefore, by Chebyshev's inequality, for all $\eta > 0$, we have

$$\mathbf{P}\{N(\epsilon) \geq \epsilon^{s-2-\eta}\} \leq C\epsilon^{\eta}.$$

Now fix $n \in \mathbb{N}$ and set $\epsilon = 2^{-n}$. Since the sequence $2^{-n\eta}$ is summable, it follows by the Borel-Cantelli lemma that a.s., there exists n_0 such that, for all $n \geq n_0$, we have $N_{2^{-n}} \leq 2^{(2-s+\eta)n}$.

Since the family (C_ϵ) is decreasing, any covering of C_ϵ is also a covering of C. The previous estimate then says that a.s., for all n large enough, it is possible to cover C with at most $2^{(2-s+\eta)n}$ balls of radius 2^{-n}. Therefore, the box dimension of C is a.s. not greater than $2 - s + \eta$. Letting $\eta \to 0+$, we obtain, with probability 1, that

$$\dim_H C \leq \dim_{box} C \leq 2 - s.$$

We have shown that if conditions (i) and (ii) hold, then a.s., the Hausdorff dimension of C is less than or equal to $2-s$. We omit the proof of the other part of the theorem, that conditions (i) and (iii) hold, then the Hausdorff dimension of C is greater than or equal to $2 - s$ with strictly positive probability. □

A.3 Heuristic derivation of σ

In chapter 6, it was conjectured that there exists a constant $\sigma > 0$ such that $n^{-\sigma}Y_n$ converges in distribution to that of a stable random variable, where Y_n is the n-th bridge height of an infinite length SAW in $\mathbb{H}$. Here we give a derivation for the conjecture that $\sigma = 4/3$. The argument given is originally due to Tom Kennedy via private communication.

To begin, let $B_h(z)$ be the generating function for bridges in a strip of height height h starting at 0. That is, if we let $h(\omega)$ denote the height of $\omega \in \mathcal{B}$. That is,

$$B_h(z) = \sum_{\omega \in \mathcal{B}} z^{|\omega|} 1(h(\omega) = h).$$

Based on [LSW2002], $B_h(z)$ should decay like $h^{-1/4}$. The argument is essentially that if we constrain the bridge to end at a fixed point along the upper boundary, $B_h(z)$ decays like h^{-2b}, where $b = 5/8$. The number of endpoints that contribute to $B_h(z)$ is of order h, and we are left with a decay of $B_h(z) \asymp h^{-1/4}$.

We will derive a relationship between $B_h(z)$ and the cumulative distribution of the height of an irreducible bridge. We run an i.i.d. sequence of irreducible bridges until the total height of the concatenation strictly exceeds L. This happens with probability 1, so

$$\begin{aligned}
1 &= \sum_{n=1}^{\infty} \sum_{\omega^1,\ldots,\omega^n \in \mathcal{I}} \beta^{-\sum_{j=1}^{n} |\omega^j|} 1(\sum_{j=1}^{n} h(\omega^j) > L) 1(\sum_{j=1}^{n-1} h(\omega^j) \le L) \\
&= \sum_{h=0}^{L} \sum_{n=1}^{\infty} \sum_{\omega^1,\ldots,\omega^n \in \mathcal{I}} \beta^{-\sum_{j=1}^{n} |\omega^j|} 1(\sum_{j=1}^{n-1} h(\omega^j) = h) 1(\sum_{j=1}^{n} h(\omega^j) > L) \\
&= \sum_{h=0}^{L} \sum_{n=1}^{\infty} \sum_{\omega^1,\ldots,\omega^{n-1} \in \mathcal{I}} \beta^{-\sum_{j=1}^{n-1} |\omega^j|} 1(\sum_{j=1}^{n-1} h(\omega^j) = h) \sum_{\omega^n \in \mathcal{I}} \beta^{-|\omega^n|} 1(h + h(\omega^n) > L).
\end{aligned}$$

Next we use

$$\sum_{\omega \in \mathcal{I}} \beta^{-|\omega|} 1(h + h(\omega) > L) = \mathbf{P}(h(\omega) > L - h),$$

where $\mathbf{P}$ denotes the probability measure on irreducible bridges which assigns probability $\beta^{-|\omega|}$ to each $\omega \in \mathcal{I}$. Now, we have

$$\begin{aligned}
1 &= \sum_{h=0}^{L} \sum_{n=1}^{\infty} \sum_{\omega^1,\ldots,\omega^{n-1} \in \mathcal{I}} \beta^{-\sum_{j=1}^{n-1} |\omega^j|} 1(\sum_{j=1}^{n-1} h(\omega^j) = h) \mathbf{P}(h(\omega) > L - h) \\
&= \sum_{h=0}^{L} B_h(\beta^{-1}) \mathbf{P}(h(\omega) > L - h).
\end{aligned}$$

Let us now assume that $B_h(\beta^{-1}) \asymp h^{-1/4}$ and $\mathbf{P}(h(\omega) > h) \asymp h^{-p}$ for some power p. We will split the above identity into two sums: one from 0 to $L/2 - 1$, and one from $L/2$ to L. In the first sum, $L - h/2$ is at the least $L/2$, and so $\mathbf{P}(h(\omega) > L - h)$ is (up to multiplicative constants) L^{-p}. So the first sum behaves like

$$\sum_{h=0}^{L/2-1} h^{-1/4} L^{-p} \asymp L^{-p+3/4}.$$

In the second sum, $h \geq L/2$, and so $B_h(\beta^{-1})$ is (up to multiplicative constants), $L^{-1/4}$. So the second sum behaves like

$$\sum_{L/2}^{L} L^{-1/4}(L-h)^{-p} = \sum_{0}^{L/2} L^{-1/4} h^{-p} \asymp L^{3/4-p},$$

so both sums behave like $L^{3/4-p}$. As $L \to \infty$, the identity says that this cannot diverge or go to zero, and therefore we should have $p = 3/4$.

In conclusion, $\mathbf{P}(h(\omega) > h)$ decays like

$$\mathbf{P}(h(\omega) > h) \asymp h^{-3/4}.$$

This tells us which stable process the sum of n irreducible bridges converges to in distribution. Let Y_n denote the n-th bridge height. We want to find σ so that Y_n grows like n^σ. The *cdf* $F(h)$ of the irreducible bridge heights converges to 1 like $1 - h^{-3/4}$ as $h \to \infty$. If there are n irreducible bridges, the largest one will roughly have height h, so $F(h) \approx 1 - 1/n$. Thus, $h \sim \text{const}n^{4/3}$, i.e. $\sigma = 4/3$.

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