On Dissipation Function of Ocean Waves due to Whitecapping.

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March 28, 2010
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Waves forecasting.
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Scheme of scales

\[ k = \frac{2\pi}{\lambda} \]

\[ \gamma \]

inverse cascade

pumping

direct cascade

damping

\[ k_p1, k_p2, k_d \]

k=\[2\pi/\lambda\]
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Why it is important?
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**Purpose of wave forecasting**
Kinetic equation

The pair correlation function for excitations $N_k$ obeys the kinetic equation (Nordheim, 1929; Hasselmann, 1962; Zakharov, 1966)

$$\frac{\partial N_k}{\partial t} = st(N, N, N) + f_p(k) - f_d(k),$$

(1)

Here

$$st(N, N, N) = 4\pi \int \left| T_{k, k_1, k_2, k_3} \right|^2 \times$$

$$\times (N_{k_1}N_{k_2}N_{k_3} + N_kN_{k_2}N_{k_3} - N_kN_{k_1}N_{k_2} -$$

$$-N_kN_{k_1}N_{k_3})\delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3)\delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})d\vec{k}_1 d\vec{k}_2 d\vec{k}_3.$$  

(2)

The kinetic equation and its modifications are the base for all wave forecasting models.
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White capping.
Dissipation function.

Dissipative part of kinetic equation

\[
\frac{\partial N_k}{\partial t} = \ldots - \gamma_{k,\mu}^{\text{kin}} \omega_k N_k.
\]

If \( N_k \) is almost monochromatic (swell) we can find dependence of \( \gamma^{\text{kin}} \) on average steepness \( \mu \):

\[
\gamma^{\text{kin}}(\mu) = -\frac{\dot{N}}{\omega_p N},
\]

Here

\[
N = \int n_k d^2k.
\]
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Problem formulation

Let us consider a potential flow of an ideal fluid of infinite depth with a free surface. We use standard notations for velocity potential $\phi(\vec{r}, z, t), \vec{r} = (x, y)$; $\vec{v} = \nabla \phi$ and surface elevation $\eta(\vec{r}, t)$.

Steepness of the surface $\mu = \sqrt{\langle |\nabla \eta(\vec{r}, t)|^2 \rangle}$ — average slope of the surface.
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**Energy of the system**

Fluid flow is incompressible \( (\nabla \vec{v}) = \Delta \phi = 0 \). The total energy of the system can be presented in the following form

\[
H = T + U,
\]

Kinetic energy:

\[
T = \frac{1}{2} \int d^2r \int_{-\infty}^{\eta} (\nabla \phi)^2 \, dz, \tag{5}
\]

Potential energy due to gravity:

\[
U = \frac{1}{2} g \int \eta^2 d^2r, \tag{6}
\]

here \( g \) is the gravity acceleration.

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**Hamiltonian expansion**

It was shown by Zakharov (1966) that under these assumptions the fluid is a Hamiltonian system

\[
\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta},
\]

(7)

where \( \psi = \phi(\vec{r}, \eta(\vec{r}, t), t) \) is a velocity potential on the surface of the fluid. In order to calculate the value of \( \psi \) we have to solve the Laplace equation in the domain with varying surface \( \eta \). One can simplify the situation, using the expansion of the Hamiltonian in powers of "steepness" (here \( \Delta = \nabla^2 \) and \( \hat{k} = \sqrt{-\Delta} \))

\[
H = \frac{1}{2} \int \left( g \eta^2 + \psi \hat{k} \psi \right) d^2r + \\
+ \frac{1}{2} \int \eta \left[ |\nabla \psi|^2 - (\hat{k} \psi)^2 \right] d^2r + \\
+ \frac{1}{2} \int \eta (\hat{k} \psi) \left[ \hat{k} (\eta (\hat{k} \psi)) + \eta \Delta \psi \right] d^2r.
\]

(8)
Dynamical equations

In this case dynamical equations acquire the following form

\[ \dot{\eta} = \hat{k}\psi - (\nabla(\eta \nabla \psi)) - \hat{k}[\eta\hat{k}\psi] + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \frac{1}{2}\Delta[\eta^2\hat{k}\psi] + \frac{1}{2}\hat{k}[\eta^2\Delta\psi] - D_{\vec{r}}, \]

\[ \dot{\psi} = -g\eta - \frac{1}{2} \left[ (\nabla\psi)^2 - (\hat{k}\psi)^2 \right] - \left[ \hat{k}\psi\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi \right] - D_{\vec{r}} + F_{\vec{r}}. \]

Here $D_{\vec{r}}$ is some artificial damping term used to provide dissipation at small scales; $F_{\vec{r}}$ is a pumping term corresponding to external force (having in mind wind blow, for example). Let us introduce Fourier transform

\[ \psi_{\vec{k}} = \frac{1}{2\pi} \int \psi_{\vec{r}} e^{i\vec{k}\vec{r}} d^2r, \quad \eta_{\vec{k}} = \frac{1}{2\pi} \int \eta_{\vec{r}} e^{i\vec{k}\vec{r}} d^2r. \]
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**Numerical scheme parameters**

Let us add pseudo-viscous damping in dynamical equations

\[ \dot{\eta} = \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \frac{1}{2}\Delta[\eta^2\hat{k}\psi] + \frac{1}{2}\hat{k}[\eta^2\Delta\psi] - F^{-1}[\gamma_k\eta\hat{k}], \]

\[ \dot{\psi} = -g\eta - \frac{1}{2}[(\nabla\psi)^2 - (\hat{k}\psi)^2] - [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi - F^{-1}[\gamma_k\psi\hat{k}]. \]

(10)

\[ \gamma_k = \gamma_0(k - k_d)^2, \quad k > k_d, \quad \gamma_0 = 2.86 \times 10^{-3}; \]

\[ \gamma_k = 0, \quad k \leq k_d. \]

(11)

Gravity acceleration \( g = 1 \). Simulation region \( L_x = L_y = 2\pi \) with double periodic boundary conditions. Rectangular numerical grid \( N_x = 512, \ N_y = 4096 \). Pseudo-viscous dissipation starts at \( k_d = 1024 \). Time step \( \Delta t = 4.23 \times 10^{-4} \)
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**Scheme of scales**

![Diagram showing scales with Downshift, spectral maximum, direct cascade, and damping regions.](image)

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**Initial conditions.**

Gauss-shaped spectrum, centered at $\vec{k} = (0; 100)$ with width $D = 30$.

\[
\begin{aligned}
|a_{\vec{k}}| &= A_i \exp \left(-\frac{1}{2} \frac{\left|\vec{k} - \vec{k}_0\right|^2}{D_i^2}\right), \quad \left|\vec{k} - \vec{k}_0\right| \leq 2D_i, \\
|a_{\vec{k}}| &= 10^{-12}, \quad \left|\vec{k} - \vec{k}_0\right| > 2D_i.
\end{aligned}
\] (12)
Dissipation function. The first experiment.

\[ \gamma_{\text{kin}} \]

\[ \mu \]

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**Proposed energy transfer mechanism. The first experiment.**

**Mechanism:**

- High steepness $\rightarrow$ wide spectrum $\rightarrow$ energy quickly delivered to the dissipation region and dissipated completely.

**Problem:**

- Weakly nonlinear model $\rightarrow$ we cannot model wavebreaking or whitecapping in details which are strongly nonlinear phenomena.
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**Model of energy transfer mechanism. The first experiment.**

Remedy for a problem:

- We don’t need to know wavebreaking **details**, because we need to know how much energy was dissipated, instead of how in details it was dissipated.

- Multiple harmonics generation describes spectrum widening during early stage of wavebreaking and whitecapping. This nonresonant mechanism is taken into account in our dynamic equations.

- Due to the universal mechanism of the spectrum widening we can check our results in the fully nonlinear 2D-model, result should be the same.
Fully nonlinear 2D experiment.

Suppose that incompressible fluid covers the domain

$$-\infty < y < \eta(x, t).$$

The flow is potential and incompressible, hence $v = \nabla \phi, \nabla v = 0, \Delta \phi = 0$.

$$H = T + U,$$

Kinetic energy:

$$T = \frac{1}{2} \int dxdy \int_{-\infty}^{\eta} (\nabla \phi)^2 dy,$$  \hspace{1cm} (13)

Potential energy due to gravity:

$$U = \frac{1}{2} g \int \eta^2 dx$$  \hspace{1cm} (14)
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Hamiltonian equations.

\[ \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \]

(15)

One can perform the conformal transformation to map the domain that is filled with fluid,

\[-\infty < x < +\infty, \quad -\infty < y < \eta(x, t), \quad z = x + iy,\]

in \( z \)-plane to the lower half-plane

\[-\infty < u < +\infty, \quad -\infty < v < 0, \quad w = u + iv,\]

in \( w \)-plane.
Hilbert transformation.

Now, the shape of surface $\eta(x, t)$ is presented by parametric equations

$$y = y(u, t), \quad x = x(u, t).$$

where $x(u, t)$ and $y(u, t)$ are related through Hilbert transformation

$$y(u, t) = \hat{H}(x(u, t) - u), \quad x(u, t) = u - \hat{H}(y(u, t)).$$

$$\hat{H}(f(u)) = \frac{1}{\pi v.p.} \int_{-\infty}^{+\infty} \frac{f(u')du}{u' - u}$$

Kinetic energy term in new variables:

$$\frac{\partial \Phi}{\partial v} \bigg|_{v=0} = - \frac{\partial}{\partial u} \hat{H}\Phi.$$
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\[ \gamma \]

\[ k_p \]

\[ k_d \]

E

damping

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Dissipation function. The second experiment.

\[ \gamma_{\text{kin}} \]

\[ \mu \]

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Dissipation function. Both experiments.

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\textbf{Waves forecasting models.}

\[ \gamma_{\vec{k}} = C_{ds} \tilde{\omega} \frac{\vec{k}}{k} \left( (1 - \delta) + \delta \frac{\vec{k}}{k} \right) \left( \frac{\tilde{S}}{\tilde{S}_{pm}} \right)^p \]  

where \( k \) and \( \omega \) are the wave number and frequency, tilde denotes mean value; \( C_{ds}, \delta \) and \( p \) are tunable coefficients; \( S = \tilde{k} \sqrt{H} \) is the overall steepness; \( \tilde{S}_{PM} = (3.02 \times 10^{-3})^{1/2} \) is the value of \( \tilde{S} \) for the Pierson-Moscowitz spectrum (note that the characteristic steepness is \( \mu = \sqrt{2}S \)). The values of the tunable coefficients for the \( WAM3 \) case are:

\[ C_{ds} = 2.35 \times 10^{-5}, \quad \delta = 0, \quad p = 4 \]  

(17)

and for the \( WAM4 \) case are:

\[ C_{ds} = 4.09 \times 10^{-5}, \quad \delta = 0.5, \quad p = 4 \]  

(18)
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Dissipation function. Comparison with waves forecasting models.

Experiment with $k_0 = 100$.

- 2D fully nonlinear experiment
- 3D, from $70T_0$ to $80T_0$
- 3D, from $80T_0$ to $90T_0$
- 3D, from $90T_0$ to $100T_0$

$2.579\mu^4 = WAM3$ for $\delta(k)$ wave packet.

$4.481\mu^4 = WAM4$ for $\delta(k)$ wave packet.
Dissipation function. Exponential fit.

Experiment with $k_0 = 100$.

\begin{align*}
3D \text{ and } 2D \text{ data} & \quad 2.327 \times 10^{-7} \exp(66.657 \mu) \\
& \quad 3.201 \times 10^{-8} \exp(73.077 \mu)
\end{align*}
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**Dissipation function. Exponential fit. Low $\mu$.**

Experiment with $k_0 = 100$.

![Graph showing dissipation function with exponential fit](image)

$\gamma_{kin}$ vs $\mu$

- 3D and 2D data
- $2.327 \times 10^{-7} \exp(66.657\mu)$
- $3.201 \times 10^{-8} \exp(73.077\mu)$
Results.

• Performed simulation of the gravity waves decaying turbulence in 2D fully-nonlinear and 3D weakly-nonlinear models.

• Obtained dependence of the dissipation function on average steepness.

• Demonstrated threshold-like character of the dissipation due to whitecapping.

• Results are significantly different with respect to wave-forecasting models terms.