Graded Problems from Homework 8.

Section 3.7

#26 Find the equation of the tangent line to the curve

$$y = \frac{x}{y+a} \quad \text{at} \quad (0,0).$$

The equation of the tangent line passes through the point $p = (0,0)$ with slope $m = \left.\frac{dy}{dx}\right|_{(0,0)}$.

To find $\frac{dy}{dx}$ we implicitly differentiate.

$$y = \frac{x}{y+a}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{y+a}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(y+a)\cdot\frac{d}{dx}x - x\cdot\frac{d}{dx}(y+a)}{(y+a)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y+a - x\cdot\frac{dy}{dx}}{(y+a)^2}$$

Get rid of dinominator

$$(y+a)^2 \cdot \frac{dy}{dx} = y+a - x\cdot\frac{dy}{dx}$$

Get all $\frac{dy}{dx}$'s on one side

$$(y+a)^2 \frac{dy}{dx} + x\frac{dy}{dx} = y+a$$

Factor

$$\left[(y+a)^2 + x\right]\frac{dy}{dx} = y+a$$

Solve

$$\frac{dy}{dx} = \frac{y+a}{(y+a)^2+x}$$

Evaluates

$$m = \left.\frac{dy}{dx}\right|_{(0,0)} = \frac{0+a}{(0+a)^2+0} = \frac{1}{a}$$

Line

$$y - 0 = \frac{1}{a}(x-0)$$

Tangent line $\boxed{y = \frac{1}{a}x}$

There is an easier way...

Consider the equation

$$y = \frac{x}{y+a}$$

rewrite it

$$y(y+a) = x$$

$$y^2 + ay = x$$

Differentiate

$$2y\cdot\frac{dy}{dx} + a\cdot\frac{dy}{dx} = 1$$

$$(2y+a)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y+a}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(0,0)} = \frac{1}{a}$$

Just like before!

Section 3.8

#13.

Show that $\sinh(-x) = -\sinh(x)$.

Calculate:

$$\sinh(-x) = \frac{1}{2}\left(e^{-x} - e^{-(-x)}\right)$$

$$= \frac{1}{2}\left(e^{-x} - e^{x}\right)$$

$$= \frac{-1}{2}\left(e^{x} - e^{-x}\right) \leftarrow \text{factor out } -1.$$

$$= -\sinh(x)$$ ✓

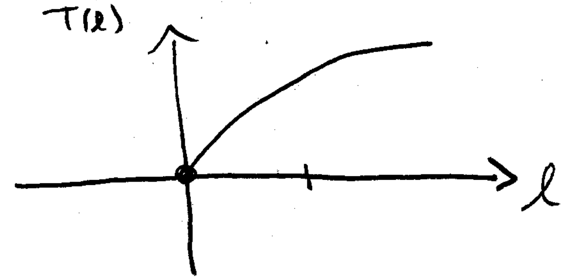
Section 3.9

#22. The period T of a pendulum of length l is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$ where g is gravity.

a) Show that if the length of the pendulum goes from l to $l+\Delta l$, then the change in period $\Delta T \approx \frac{T}{2l}\Delta l$.

To solve this we note that the period T is a function of the length



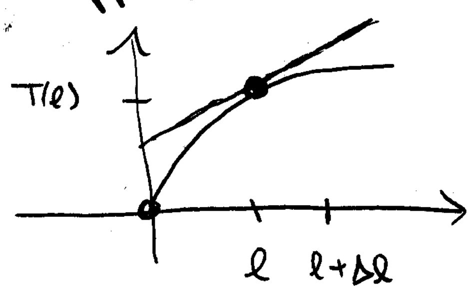
$$T(l) = 2\pi\sqrt{\frac{l}{g}}$$

We can calculate the derivative of T

$$\frac{dT}{dl} = 2\pi \cdot \frac{d}{dl}\sqrt{\frac{l}{g}} = \cancel{2}\pi \cdot \frac{1}{\cancel{2}\sqrt{\frac{l}{g}}}\,\frac{d}{dl}\left(\frac{l}{g}\right) = \pi \cdot \sqrt{\frac{g}{l}} \cdot \frac{1}{g}$$

$$\Rightarrow \quad \frac{dT}{dl} = \frac{\pi}{\sqrt{l}\cdot\sqrt{g}}$$

To approximate $T(l+\Delta l)$ we use the tangent line to T at l.



$T(l+\Delta l) \cong$ The tangent line to T at l evaluated at $l+\Delta l$.

The tangent line to T at l $= T(l) + T'(l)(x - l)$

If we evaluate this at $x = l+\Delta l$, we see that

$$T(l+\Delta l) \cong T(l) + T'(l)(l+\Delta l - l)$$
$$= T(l) + T'(l)\cdot\Delta l$$

we Know the derivative of T

$$T'(l) = \frac{\pi}{\sqrt{l}\cdot\sqrt{g}}$$

So

$$T(l+\Delta l) \cong T(l) + \frac{\pi}{\sqrt{l}\cdot\sqrt{g}}\cdot\Delta l .$$

From this we can calculate the change in T

$$\Delta T = T(l+\Delta l) - T(l)$$

$$\cong \frac{\pi}{\sqrt{l}\cdot\sqrt{g}}\cdot\Delta l$$

$$= \frac{1}{2\sqrt{l}}\,2\pi\frac{\sqrt{l}}{\sqrt{g}}\cdot\frac{1}{\sqrt{l}}\,\Delta l$$

$$= \frac{1}{2l}\,T\,\Delta l \quad \checkmark$$

b) If the length of the pendulum increases by 2%, by what % does the period change.

if the length increases by 2% $\Rightarrow$ $\frac{\Delta l}{l} = 0.02$

The percent change in the period is then $\frac{\Delta T}{T}$,

we saw in part a that

$$\Delta T \cong \frac{T}{2\ell} \Delta \ell$$

$$\Rightarrow \quad \frac{\Delta T}{T} \cong \frac{1}{2} \frac{\Delta \ell}{\ell} = \frac{1}{2} \cdot 0.02 = 0.01$$

$\Rightarrow$ $\frac{\Delta T}{T}$ is 1%.