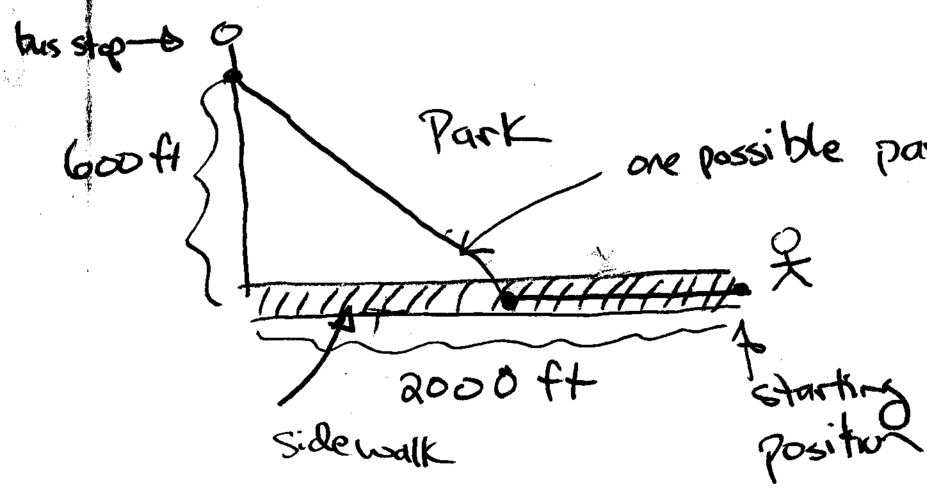
Ex

Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2000 ft West and 600 ft North of her starting position. Alaina can walk West along the edge of the park (on the sidewalk) at a speed of 6 ft/sec. She can also travel through the grass in the park, but only at a rate of 4 ft/sec. What path will get her to the bus stop the fastest?

Let x be the distance Alaina travels on the sidewalk.



If x is the distance she travels on the sidewalk the domain is $0 \le x \le 2000$.

Since she wants to do this as quickly as possible, we want to optimize the total time of her journey.

$$\text{Total Time} = \begin{matrix}\text{Time on}\\ \text{sidewalk}\end{matrix} + \begin{matrix}\text{Time in}\\ \text{park}\end{matrix}$$

All we know are distances and rates...

But there is a basic formula:

$$\text{Distance} = (\text{Rate})(\text{Time})$$

Thus we have that

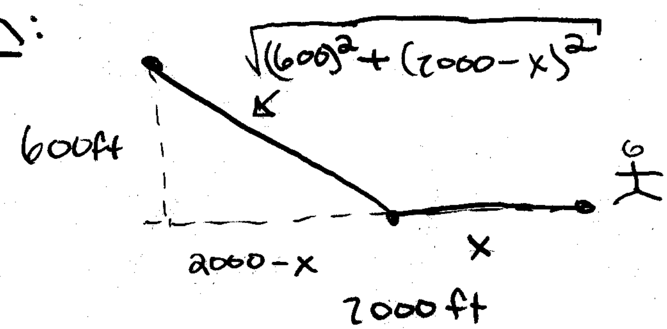
$$\text{Total Time} = \begin{matrix}\text{Time on}\\ \text{side walk}\end{matrix} + \begin{matrix}\text{Time in}\\ \text{park}\end{matrix}$$

$$= \frac{\text{Distance on sidewalk}}{\text{rate on sidewalk}} + \frac{\text{Distance in park}}{\text{rate in park}}$$

$$= \frac{x}{6} + \frac{\sqrt{(600)^2 + (2000-x)^2}}{4}$$

$$\Rightarrow \text{Total Time} = T(x) = \frac{x}{6} + \frac{\sqrt{(600)^2 + (2000-x)^2}}{4}$$

Recall the path:



Now we optimize this function on $[0, 2000]$.

$$\frac{dT(x)}{dx} = \frac{1}{6} + \frac{1}{4}\cdot\frac{1}{2\sqrt{(600)^2+(2000-x)^2}}\cdot\frac{d}{dx}\left((600)^2+(2000-x)^2\right)$$

$$= \frac{1}{6} + \frac{2(2000-x)(-1)}{8\sqrt{(600)^2+(2000-x)^2}}$$

To find the critical points we set this function equal to zero.

$$\frac{dT}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{6} - \frac{(2000-x)}{4\sqrt{(600)^2+(2000-x)^2}} = 0$$

$$\Rightarrow \quad \frac{1}{6} = \frac{2000-x}{4\sqrt{(600)^2+(2000-x)^2}}$$

cross mult.

$$\Rightarrow \quad 4\sqrt{(600)^2+(2000-x)^2} = 6(2000-x)$$

square

$$16\left((600)^2+(2000-x)^2\right) = 36(2000-x)^2$$

$$\Rightarrow \quad 16(600)^2+16(2000-x)^2 = 36(2000-x)^2$$

$$\Rightarrow \quad 16(600)^2 = 20(2000-x)^2$$

$$\Rightarrow \quad (2000-x) = \pm\sqrt{\frac{16(600)^2}{20}}$$

$$X = 2000 \pm \frac{4 \cdot 600}{2 \cdot \sqrt{5}}$$

only $X = 2000 - \frac{2 \cdot 600}{\sqrt{5}}$ is in the domain

$\cong 1463.34$

So if she takes the path where she stays on the side walk for 1463 ft, and then heads to the bus stop through the park, this will minimize her time.

<u>Note:</u> (Below I check the endpoints to compare!)

If she spends no time on the sidewalk,

$$T(0) = \frac{0}{6} + \frac{\sqrt{(600)^2 + (2000-0)^2}}{4} \cong 522 \text{ seconds.}$$

If she takes the sidewalk all the way to the end,

$$T(2000) = \frac{2000}{6} + \frac{\sqrt{(600)^2 + (2000-2000)^2}}{4}$$

$$\cong 333.3 + 150 = 483.3 \text{ seconds}$$

At the critical point

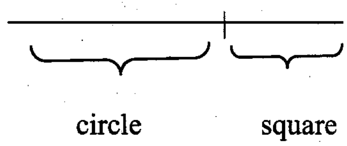
$$T(1463) \cong \frac{1463}{6} + \frac{\sqrt{(600)^2 + (2000-1463)^2}}{4}$$

$$= 243.8 + 201.3$$

$$= 445.1 \text{ seconds}$$ ← The minimum!

OPTIMIZATION – PART 1 NAME Key

1. A wire of length 12 inches can be bent into a circle, a square, or cut to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be a minimum? A maximum?



Let x be the amount of wire for the circle. This means that $12-x$ is the amount of wire for the square. Thus

$$x = \text{circumference of circle} = 2\pi r \quad (r = \text{radius of circle})$$

$$12-x = \text{Perimeter of square} = 4s \quad (s = \text{side length})$$

We wish to optimize area.

$$\text{Area} = \text{Area of Circle} + \text{Area of square} = \pi r^2 + s^2$$

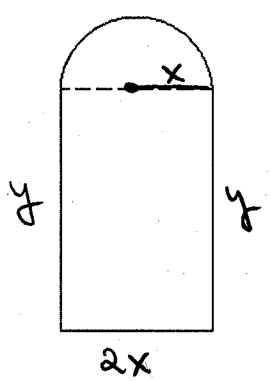
$$A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{12-x}{4}\right)^2$$

$$\Rightarrow A(x) = \frac{x^2}{4\pi} + \frac{(12-x)^2}{16}$$

$$A'(x) = \frac{x}{2\pi} + \frac{2(12-x)}{16}\cdot(-1)$$

(over see page II)

2. A window consisting of a rectangle topped by a semicircle is to have an outer perimeter P. Find the radius of the semicircle if the area of the window is to be a maximum.



Let x be the radius of the semi-circle.
Let y be the height of the (rectangular part of) window.

$$P = \text{Perimeter} = 2x + 2y + \tfrac{1}{2}(\text{circumference of circle})$$
$$= 2x + 2y + \tfrac{1}{2}(2\pi x)$$

$$\Rightarrow P = (2+\pi)x + 2y$$

$$\Rightarrow 2y = P - (2+\pi)x$$

$$y = \frac{P-(2+\pi)x}{2}$$

We wish to optimize area.

$$\text{Area} = \text{Area of semi-circle} + \text{Area of rectangle}$$

$$= \tfrac{1}{2}\pi x^2 + 2x\cdot y$$

$$\Rightarrow A(x) = \frac{\pi}{2}x^2 + \cancel{2}x\cdot\frac{P-(2+\pi)x}{\cancel{2}}$$

(over see page III)

To find the critical points, we set

$$A'(x) = 0$$

$$\Rightarrow \quad \frac{x}{2\pi} - \frac{(12-x)}{8} = 0$$

multiply by 8π

$$8\pi\left(\frac{x}{2\pi} - \frac{(12-x)}{8}\right) = 8\pi(0)$$

$$\Rightarrow \quad 4x - \pi(12-x) = 0$$

$$(4+\pi)x = 12\pi$$

$$x = \frac{12\pi}{4+\pi} \cong 5.279$$

Let's Test the critical points and endpoints.

Just a square $\Rightarrow$ $A(0) = \dfrac{0^2}{4\pi} + \dfrac{12^2}{16} = 9$

Just a circle $\Rightarrow$ $A(12) = \dfrac{(12)^2}{4\pi} + \dfrac{(12-12)^2}{16} \cong 11.46$

critical point $\Rightarrow$ $A(5.279) = \dfrac{(5.279)^2}{4\pi} + \dfrac{(12-5.279)^2}{16}$

$$\cong 2.218 + 2.823$$

$$= 5.041$$

So we must use 5.279 inches to minimize area.

and we must use 12 inches to maximize area.

So, as a function of x,

$$A(x) = \frac{\pi}{2}x^2 + x\left(P - (2+\pi)x\right)$$

Now optimize

$$A'(x) = \pi x + P - (2+\pi)x - (2+\pi)x$$

$$= (-4-\pi)x + P$$

The critical points are when $A'(x) = 0$

ie. $(-4-\pi)x + P = 0$

$$(4+\pi)x = P$$

$$x = \frac{P}{4+\pi}$$

Let's Test the critical point and endpoints.

If $x=0$, then $A(0)=0$.
The biggest x can be is when $y=0$.
Then $P = (2+\pi)x$

ie. $x = \frac{P}{2+\pi}$

$$A\left(\frac{P}{2+\pi}\right) = \frac{\pi}{2}\left(\frac{P}{2+\pi}\right)^2 + \frac{P}{2+\pi}\left(\cancelto{0}{P - (2+\pi)\cdot\frac{P}{2+\pi}}\right)$$

$$= \frac{\pi}{2(2+\pi)^2}\cdot P^2$$

$$\cong 0.06\,P^2$$

and if $x = \frac{p}{4+\pi}$, then

$$A\left(\frac{p}{4+\pi}\right) = \frac{\pi}{2}\left(\frac{p}{4+\pi}\right)^2 + \frac{p}{4+\pi}\cdot\left(p - (2+\pi)\cdot\frac{p}{4+\pi}\right)$$

$$= \frac{\pi}{2(4+\pi)^2}p^2 + \frac{1}{4+\pi}\left(\frac{4+\pi-2-\pi}{4+\pi}\right)p^2$$

$$= \left(\frac{\pi}{2}+2\right)\frac{p^2}{(4+\pi)^2}$$

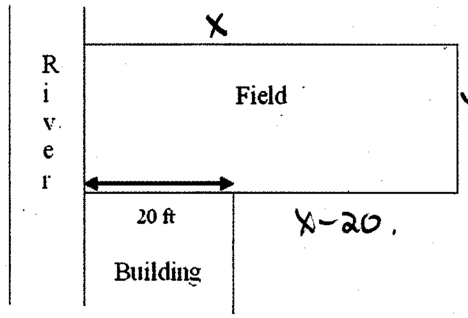
$$= 0.07\,p^2$$

This is the maximum area

and the radius is $\frac{p}{4+\pi}$.

V

3. A rectangular field as shown is to be bounded by a fence. Find the dimensions of the field with maximum area that can be enclosed with 1000 feet of fencing. You can assume that fencing is not needed along the river and building.



Let x be the length and y be the width of the field.

Then

$$1000 = x + y + (x-20)$$

$$\Rightarrow y = 1020 - 2x$$

$$\boxed{\begin{array}{l}\text{Dimensions} \\ x = 255 \\ y = 510\end{array}}$$

We wish to optimize the area.

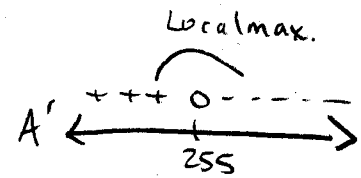
$$\text{Area} = x \cdot y$$

$$A(x) = x(1020-2x)$$

$$A'(x) = 1020 - 2x - 2x$$
$$= 1020 - 4x$$

Critical points:

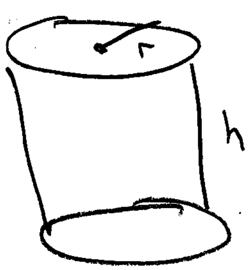
$$A'(x) = 0 \Rightarrow 1020 - 4x = 0$$
$$4x = 1020 \Rightarrow x = 255$$



$$y = 1020 - 2(255) = 510$$

4. A company manufactures cylindrical barrels to store nuclear waste. The top and bottom of the barrels are to be made with material that costs \$10 per square foot and the rest is made with material that costs \$8 per square foot. If each barrel is to hold 5 cubic feet, find the dimensions of the barrel that will minimize the total cost.

Let r be the radius of the barrel and h be the height.



We wish to optimize cost.

$$\text{Cost} = \text{cost of Top + Bottom} + \text{cost of shell}$$

$$= 10(2\pi r^2) + 8(2\pi r h)$$

$$\text{Cost} = 20\pi r^2 + 16\pi r h \quad \leftarrow \text{Problem! 2 variables.}$$

We know that

$$5\,(ft)^3 = \text{Volume} = \pi r^2 h$$

$$\Rightarrow h = \frac{5}{\pi r^2} \quad \Rightarrow \quad \text{Cost} = 20\pi r^2 + 16\pi r\left(\frac{5}{\pi r^2}\right)$$

over

Thus we optimize

$$\text{Cost} = C(r) = 20\pi r^2 + \frac{80}{r}$$

$$C'(r) = 40\pi r - \frac{80}{r^2}$$

$$C'(r) = 0 \Rightarrow 40\pi r - \frac{80}{r^2} = 0$$

$$40\pi r = \frac{80}{r^2}$$

$$\Rightarrow r^3 = \frac{2}{\pi}$$

$$\Rightarrow r = \sqrt[3]{\frac{2}{\pi}}$$

Since $5 = \pi r^2 h$

$$\Rightarrow h = \frac{5}{\pi r^2} = \frac{5}{\pi\left(\frac{2}{\pi}\right)^{2/3}}$$

These are the dimensions of the barrel That minimizes cost.