Worksheet:

The Product and Quotient Rule

1. $f(x) = (1+\sqrt{x})\cdot(x^3)$

$$\frac{d}{dx}f(x) = \frac{d}{dx}(1+\sqrt{x})\cdot x^3 + (1+\sqrt{x})\cdot\frac{d}{dx}x^3$$

$$= \frac{1}{2\sqrt{x}}\cdot x^3 + (1+\sqrt{x})\cdot(3x^2)$$

$$= \frac{1}{2}x^{5/2} + 3x^2 + 3x^{5/2}$$

$$= \frac{7}{2}x^{5/2} + 3x^2$$

(also)

$$f(x) = (1+\sqrt{x})(x^3) = x^3 + x^{7/2}$$

$$f'(x) = 3x^2 + \frac{7}{2}x^{5/2}$$

2. $g(t) = \left(\frac{2}{t} + t^5\right)(t^3+1)$

$$= 2t^2 + \frac{2}{t} + t^8 + t^5$$

$$\Rightarrow \frac{d}{dt}g(t) = \frac{d}{dt}(2t^2) + \frac{d}{dt}(2t^{-1}) + \frac{d}{dt}(t^8) + \frac{d}{dt}(t^5)$$

$$= 4t - 2t^{-2} + 8t^7 + 5t^4$$

<u>also</u>

$$g(t) = \left(\frac{2}{t} + t^5\right)(t^3+1)$$

$$\frac{d}{dt}g(t) = \frac{d}{dt}\left(\frac{2}{t} + t^5\right)\cdot(t^3+1) + \left(\frac{2}{t} + t^5\right)\cdot\frac{d}{dt}(t^3+1)$$

$$= \left(-2t^{-2} + 5t^4\right)(t^3+1) + \left(\frac{2}{t} + t^5\right)(3t^2)$$

$$= \underline{-2t} - \underline{2t^{-2}} + \underline{5t^7} + \underline{5t^4} + \underline{6t} + \underline{3t^7}$$

$$= 4t - 2t^{-2} + 8t^7 + 5t^4$$

3. $h(y) = \frac{1}{y^3+2y+1}$

$$\frac{d}{dy}h(y) = \frac{(y^3+2y+1)\cdot \cancel{\frac{d}{dy}(1)}^{\,0} - (1)\cdot\frac{d}{dy}(y^3+2y+1)}{(y^3+2y+1)^2}$$

$$= \frac{-3y^2 - 2}{(y^3+2y+1)^2}$$

4. $y = \dfrac{1}{x+\sqrt{x}}$

$$\frac{dy}{dx} = \frac{(x+\sqrt{x})\cdot \cancel{\frac{d}{dx}(1)}^{0} - (1)\cdot\frac{d}{dx}(x+\sqrt{x})}{(x+\sqrt{x})^2}$$

$$= \frac{-1-\frac{1}{2\sqrt{x}}}{(x+\sqrt{x})^2}$$

$$= -\frac{2\sqrt{x}+1}{2\sqrt{x}\,(x+\sqrt{x})^2}$$

5. $y = 2^x \cdot e^x$

$$\frac{dy}{dx} = \frac{d}{dx}2^x \cdot e^x + 2^x \cdot \frac{d}{dx}e^x$$

$$= \ln(2)\cdot 2^x e^x + 2^x \cdot e^x$$

$$= (1+\ln(2))\,2^x e^x$$

6. $$g(z) = \frac{z^2+1}{z^3-5}$$

$$\frac{d}{dz}g(z) = \frac{(z^3-5)\cdot\frac{d}{dz}(z^2+1) - (z^2+1)\frac{d}{dz}(z^3-5)}{(z^3-5)^2}$$

$$= \frac{(z^3-5)(2z) - (z^2+1)(3z^2)}{(z^3-5)^2}$$

$$= \frac{2z^4 - 10z - 3z^4 - 3z^2}{(z^3-5)^2}$$

$$= \frac{-z^4 - 3z^2 - 10z}{(z^3-5)^2}$$

7. $$y = \frac{\sqrt{x}}{x^3+1}$$

$$\frac{dy}{dx} = \frac{(x^3+1)\cdot\frac{d}{dx}\sqrt{x} - \sqrt{x}\cdot\frac{d}{dx}(x^3+1)}{(x^3+1)^2}$$

$$= \frac{(x^3+1)\frac{1}{2\sqrt{x}} - \sqrt{x}\,(3x^2)}{(x^3+1)^2}$$

$$= \frac{\frac{1}{2}x^{5/2} + \frac{1}{2\sqrt{x}} - 3x^{5/2}}{(x^3+1)^2} = \frac{-\frac{5}{2}x^{5/2} + \frac{1}{2\sqrt{x}}}{(x^3+1)^2}$$

8. $$z = \frac{t^2}{(t-4)(2-t^3)}$$

$$\frac{dz}{dt} = \frac{(t-4)(2-t^3)\cdot\frac{d}{dt}(t^2) - t^2\cdot\frac{d}{dt}\Big((t-4)(2-t^3)\Big)}{\Big((t-4)(2-t^3)\Big)^2}$$

$$= \frac{(t-4)(2-t^3)2t - t^2\left[\frac{d}{dt}(t-4)\cdot(2-t^3) + (t-4)\cdot\frac{d}{dt}(2-t^3\right.}{(t-4)^2(2-t^3)^2}$$

$$= \frac{(2t-t^4-8+4t^3)(2t) - t^2\left[(2-t^3)+(t-4)(-3\right.}{(t-4)^2(2-t^3)^2}$$

$$= \frac{4t^2 - 2t^5 - 16t + 8t^4 - 2t^2 + t^5 + 3t^4(t-4)}{(t-4)^2(2-t^3)^2}$$

$$= \frac{-t^5 + 8t^4 + 2t^2 - 16t + 3t^5 - 12t^4}{(t-4)^2(2-t^3)^2}$$

$$= \frac{2t^5 - 4t^4 + 2t^2 - 16t}{(t-4)^2(2-t^3)^2}$$

9. $h(x) = \dfrac{(x^3+1)\sqrt{x}}{x^2} = \dfrac{x^{7/2} + x^{1/2}}{x^2} = x^{3/2} + x^{-3/2}$

$\dfrac{d}{dx} h(x) = \dfrac{3}{2} x^{1/2} - \dfrac{3}{2} x^{-5/2}$

10. $y(m) = \dfrac{(e^m)\cdot(\sqrt[3]{m})}{m^2+3}$

$\dfrac{dy}{dm} = \dfrac{(m^2+3)\cdot\frac{d}{dm}\left(e^m\sqrt[3]{m}\right) - e^m\sqrt[3]{m}\cdot\frac{d}{dm}(m^2+3)}{(m^2+3)^2}$

$= \dfrac{(m^2+3)\left(e^m\sqrt[3]{m} + e^m\cdot\frac{1}{3}m^{-2/3}\right) - e^m\sqrt[3]{m}\,(2m)}{(m^2+3)^2}$

$= \dfrac{e^m\left(m^{\frac{7}{3}} + 3m^{1/3} - \frac{1}{3}m^{-4/3} - 2m^{4/3}\right)}{(m^2+3)^2}$

$= \dfrac{e^m m^{4/3}\left(m^2 - 2m + 3 - \frac{1}{3m}\right)}{(m^2+3)^2}$

11. $g(x) = (x+\sqrt{x})\cdot 3^x$

$$\frac{d}{dx}g(x) = \frac{d}{dx}(x+\sqrt{x})\cdot 3^x + (x+\sqrt{x})\cdot \frac{d}{dx}3^x$$

$$= \left(1+\frac{1}{2\sqrt{x}}\right)\cdot 3^x + (x+\sqrt{x})\cdot \ln(3)\cdot 3^x$$

$$= 3^x\left(\ln(3)\,x + \ln(3)\sqrt{x} + 1 + \frac{1}{2\sqrt{x}}\right)$$

12. Let $f(x) = g(x)\,h(x)$ with $g(10) = -4$, $h(10) = 560$
$g'(10) = 0$ and $h'(10) = 35$

Find $f'(10)$.

Note:

$$f'(x) = g'(x)\,h(x) + g(x)\cdot h'(x)$$

$$\Rightarrow f'(10) = g'(10)\,h(10) + g(10)\cdot h'(10)$$

$$= 0\cdot(560) + (-4)(35)$$

$$= -140$$

13. $y(x) = \dfrac{z(x)}{1+x^2}$ with $z(-3) = 6$ and $z'(-3) = 15$.

Find $y'(-3)$.

Note:

$$y'(x) = \frac{(1+x^2)\,z'(x) - z(x)\,(1+x^2)'}{(1+x^2)^2}$$

$$= \frac{z'(x) + x^2 z'(x) - 2x\,z(x)}{(1+x^2)^2}$$

So

$$y'(-3) = \frac{z'(-3) + (-3)^2 z'(-3) - 2(-3)z(-3)}{\left(1+(-3)^2\right)^2}$$

$$= \frac{15 + 9\cdot 15 + 6\cdot 6}{10^2}$$

$$= \frac{150+36}{100} = 1.86$$