Summation of Glaisher- and Apéry-like Series

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Abstract

To the author's knowledge, new techniques for the summation of certain Glaisher-like series (with terms involving $\zeta(n+1)$) and Apéry-like series (with terms involving $(n!)^2/(2n)!$) are demonstrated.

1 Introduction

In 1913, Glaisher [5] gave the following sums:

$$\sum_{n=1}^{\infty} [\zeta(2n+1)-1] = \frac{1}{4}$$
$$\sum_{n=1}^{\infty} [\zeta(6n+4)-1] = \frac{1}{12}$$
$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}}\,\zeta(2n) = \frac{1}{2}$$
$$\sum_{n=1}^{\infty} \frac{1}{2^{2n+1}}\,\zeta(2n+1) = \ln 2 - \frac{1}{2}$$
$$\sum_{n=1}^{\infty} \frac{n}{16^n}\,\zeta(2n+1) = 1 - G$$

where G is Catalan's constant defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594177....$$

Additionally, Glaisher gave many other such series which the author has termed Glaisher-like, that is, series with terms involving the zeta series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{1.1}$$

In Section 2, an interesting class of Glaisher-like series involving Lucas sequences, which the author has not been able to find in the literature, will be evaluated.

In 1979, Apéry [1] proved the irrationality of $\zeta(3)$—and in the same manner, the irrationality of $\zeta(2) = \pi^2/6$—by making use of the identities

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^{n+1}}{n^3} \tag{1.2}$$

$$\zeta(2) = 3 \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n^2}. \tag{1.3}$$

(For the interested reader—van der Poorten [12] gave a very entertaining report of Apéry's proof entitled "A proof that Euler missed....") Following Apéry's proof, many series of a similar form,

$$\sum_{n}^{\infty} \frac{(n!)^2}{(2n)!} f(n) = \sum_{n}^{\infty} \binom{2n}{n}^{-1} f(n)$$

which are commonly referred to as Apéry-like series, have been considered by van der Poorten [11] [13], Leschiner [7], Lehmer [6], Zucker [14], J. Borwein and Bradley [Searching Symbolically for Apery-like formulae for values of the Riemann Zeta function] and J. Borwein, Broadhurst, and Kamnitzer [3]. Berndt and Joshi [2] in a review of Chapter 9 of Ramanujan's 2nd notebook have also recorded many similar formulae. Section 3 will present many general evaluations of Apéry-like series, of which most appear to be new.

In the Section 3, we will make free use of Oldham and Spanier's *The Fractional Calculus* [10]. In particular, we will use the following results:

$$\frac{d^{\frac{1}{2}} f}{dx^{\frac{1}{2}}} = \frac{f(0)}{\sqrt{\pi x}} + \int_0^x \frac{f^{(1)}(y)\, dy}{\sqrt{\pi(x-y)}} \tag{1.4}$$

$$\frac{d^{-\frac{1}{2}} f}{dx^{-\frac{1}{2}}} = \int_0^x \frac{f(y)\, dy}{\sqrt{\pi(x-y)}} \tag{1.5}$$

and for $n = 0, 1, 2, ...$

$$\frac{d^{\frac{1}{2}}\, x^n}{dx^{\frac{1}{2}}} = \frac{(n!)^2\, (4x)^n}{(2n)!\sqrt{\pi x}} \tag{1.6}$$

$$\frac{d^{-\frac{1}{2}}\, x^n}{dx^{-\frac{1}{2}}} = \frac{(n!)^2\, (4x)^{n+\frac{1}{2}}}{(2n+1)!\sqrt{\pi}}. \tag{1.7}$$

We will also make use of n-th order polylogarithm $\mathrm{Li}_n(z)$ defined recursively by repeated division of $\mathrm{Li}_1(z) = -\ln(1-z)$ by z and integration:

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{Li}_{n-1}(z)}{z}\, dz. \tag{1.8}$$

If $|z| \leq 1$, then

$$\mathrm{Li}_n(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^n}. \tag{1.9}$$

For more properties, see Lewin's *Polylogarithms and associated functions* [8].

2 Summation of Glaisher-like series

Theorem 2.1 *Let $\{U_n\}$ be a Lucas sequence defined by $U_0 = 0$, $U_1 = 1$ and*

$$U_n = P\, U_{n-1} - Q\, U_{n-2} \tag{2.1}$$

where $P, Q \in \mathbb{R}$ and $D = \sqrt{P^2 - 4Q} \neq 0$. Then

$$\begin{aligned} \sum_{n=1}^{N} U_n\, \zeta(n+1) &= \frac{1}{D}\left[\psi\left(1 - \frac{P}{2} + \frac{D}{2}\right) - \psi\left(1 - \frac{P}{2} - \frac{D}{2}\right)\right] \\ &\quad - \sum_{y=1}^{\infty} \frac{\dfrac{U_{N+1}}{y^N} - Q\,\dfrac{U_N}{y^{N+1}}}{y^2 - Py + Q} \end{aligned} \tag{2.2}$$

where $\psi(x) = \dfrac{d}{dx} \ln\Gamma(x)$ is the digamma function. If $D = 0$, then

$$\sum_{n=1}^{N} U_n\, \zeta(n+1) = \psi'\left(1 - \frac{P}{2}\right) - \sum_{y=1}^{\infty} \frac{\dfrac{U_{N+1}}{y^N} - Q\,\dfrac{U_N}{y^{N+1}}}{y^2 - Py + Q} \tag{2.3}$$

Proof Let U_n be a Lucas sequence as defined above. Then, U_n has a generating function

$$\sum_{n=1}^{\infty} U_n\, x^n = \frac{x}{1 - Px + Qx^2}. \tag{2.4}$$

Letting $x = 1/y$ in Eq. (2.3) and dividing by y, we obtain

$$\sum_{n=1}^{\infty} \frac{U_n}{y^{n+1}} = \frac{1}{y^2 - Py + Q}. \tag{2.5}$$

¿From this, it is easy to see that

$$\frac{1}{y^2-Py+Q}=\frac{U_1}{y^2}+\frac{U_2}{y^3}+\frac{U_3}{y^4}+\cdots+\frac{U_N}{y^{N+1}}+\frac{\dfrac{U_{N+1}}{y^N}-Q\,\dfrac{U_N}{y^{N+1}}}{y^2-Py+Q} \tag{2.6}$$

by division. Thus, for $D\neq 0$, the theorem immediately follows by noting that

$$\sum_{y=1}^{\infty}\frac{1}{y^2-Py+Q}=\frac{1}{D}\left[\psi\left(1-\frac{P}{2}+\frac{D}{2}\right)-\psi\left(1-\frac{P}{2}-\frac{D}{2}\right)\right], \tag{2.7}$$

which is obtained through partial fraction decomposition and the use of the identity

$$\sum_{y=1}^{N-1}\frac{1}{y-\alpha}=\psi(N-\alpha)-\psi(1-\alpha), \tag{2.8}$$

as $N\to\infty$. For $D=0$, the theorem follows by taking the limit as $D\to 0$ in (2.2) and applying the limit definition of the central derivative. ■

¿From this result, we can obtain the asymptotic representation of

$$\boxed{\sum_{n=1}^{N}U_n\,\zeta(n+1)} \tag{2.9}$$

by evaluating (2.2/2.3), taking only the nonzero terms in the infinite series on the right hand side when $N\to\infty$. Seven interesting examples are the following:

1.) Taking $P=1$ and $Q=-1$ we have the Fibonacci numbers F_n and we find that

$$\sum_{n=1}^{N}F_n\,\zeta(n+1)\sim\frac{\pi\sqrt{5}}{5}\tan\frac{\pi\sqrt{5}}{2}+F_{N+2}. \tag{2.10}$$

Making use of the identity

$$\sum_{n=1}^{N}F_n=F_{N+2}-1, \tag{2.11}$$

we have that

$$\sum_{n=1}^{\infty}F_n\,[\zeta(n+1)-1]=1+\frac{\pi\sqrt{5}}{5}\tan\frac{\pi\sqrt{5}}{2}. \tag{2.12}$$

2.) Taking $P=1$ and $Q=1$ we have the interesting result

$$\sum_{n=0}^{\infty}\big[\zeta(6n+2)+\zeta(6n+3)-\zeta(6n+5)-\zeta(6n+6)\big]=\frac{\pi\sqrt{3}}{3}-\frac{2\pi\sqrt{3}}{3\,e^{\pi\sqrt{3}}+3}-1. \tag{2.13}$$

3.) Taking $P=2$ and $Q=-1$ we have the Pell numbers P_n and find that

$$\sum_{n=1}^{N} P_n\,\zeta(n+1) \sim -\frac{1}{4}\left[P_N + P_{N+2}\left(1+\frac{1}{2^{N-1}}\right) - \pi\sqrt{2}\cot\pi\sqrt{2} - 1\right]. \tag{2.14}$$

4.) Taking $P=-2$ and $Q=1$ we have

$$\sum_{n=1}^{N} (-1)^{n+1}\,n\,\zeta(n+1) \sim \frac{\pi^2}{6} - 1 + \frac{(-1)^{N+1}(2N+1)}{4}. \tag{2.15}$$

5.) Taking $P=-1$ and $Q=-1$ we have

$$\sum_{n=1}^{\infty} (-1)^{n+1}\,F_n\,[\zeta(n+1)-1] = \frac{\pi\sqrt{5}}{5}\tan\frac{\pi\sqrt{5}}{2}. \tag{2.16}$$

6.) Taking $P=0$ and $Q=1$ we have

$$\sum_{n=0}^{N} [\zeta(4n+2)-\zeta(4n+4)] \sim \frac{\pi}{e^{2\pi}-1} + \frac{\pi}{2} - \frac{1}{2}\left[1+(-1)^N\right]. \tag{2.17}$$

7.) Taking $P=0$ and $Q=-\frac{1}{2}$ we have

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2^{n-1}} = 1 - \frac{\pi\sqrt{2}}{2}\cot\frac{\pi\sqrt{2}}{2}. \tag{2.18}$$

3 Summation of Apéry-like series

We first consider the simplest Apéry-like series:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(4x)^n &= \sqrt{\pi x}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\sum_{n=0}^{\infty} x^n = \sqrt{\pi x}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{1}{1-x} \\ &= \frac{\sqrt{1-x}+\sqrt{x}\arcsin\sqrt{x}}{(1-x)^{\frac{3}{2}}}. \end{aligned}\tag{3.1}$$

Next, for $m\in\mathbb{N}$, consider the following infinite series:

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(n+1)(n+2)\cdots(n+m)(4x)^n \\ &\qquad = \sqrt{\pi x}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\sum_{n=0}^{\infty}(n+1)(n+2)\cdots(n+m)x^n \\ &\qquad = \sqrt{\pi x}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{m!}{(1-x)^{m+1}} = -\frac{m!\sqrt{x}B_x(-\frac{1}{2},m+\frac{3}{2})}{2(1-x)^{m+\frac{3}{2}}} \\ &\qquad = \frac{-m!\sqrt{x}}{2(1-x)^{m+\frac{3}{2}}}\int^x y^{-\frac{3}{2}}(1-y)^{m+\frac{1}{2}}\,dy. \end{aligned}\tag{3.2}$$

Then, if we note that

$$\int^{x} y^{\frac{-3}{2}}(1-y)^{m+\frac{1}{2}}\,dy = \frac{\sqrt{1-x}}{\sqrt{x}}\sum_{n=0}^{m} a_{n,m}x^{n} - \frac{(2m+1)!}{2^{2m-1}(m!)^{2}}\arcsin\sqrt{x}, \tag{3.3}$$

where $a_{0,m} = -2$ and

$$\left(\frac{1}{2}-n\right)a_{n,m} + (n-1)a_{n-1,m} = \binom{m+1}{n}(-1)^{n+1}, \tag{3.4}$$

we obtain the result

$$\sum_{n=0}^{\infty}\frac{(n!)^{2}}{(2n)!}(n+1)(n+2)\cdots(n+m)(4x)^{n}$$
$$= \frac{\sqrt{x}}{(1-x)^{m+\frac{3}{2}}}\frac{(2m+1)!}{2^{2m}m!}\arcsin\sqrt{x} - \frac{m!}{2(1-x)^{m+1}}\sum_{n=0}^{m} a_{n,m}x^{n}\,. \tag{3.5}$$

Now since 1 and $\{(n+1)(n+2)\cdots(n+m) : \; m \in \mathbb{N}\}$ form a basis for the vector space of all polynomials, we can find the closed form representation of any series of the type

$$\boxed{\sum_{n=0}^{\infty}\frac{(n!)^{2}}{(2n)!}\left(n^{m_1}x_1^{n} + n^{m_2}x_2^{n} + \cdots + n^{m_k}x_k^{n}\right)} \tag{I}$$

where $m_i \in \mathbb{N}\cup\{0\}$. Examples:

$$\sum_{n=0}^{\infty}\frac{(n!)^{2}}{(2n)!} = \frac{2\pi\sqrt{3}}{27} + \frac{4}{3} \tag{3.6}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^{2}\,n}{(2n)!} = \frac{2\pi\sqrt{3}}{27} + \frac{2}{3} \tag{3.7}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^{2}\,n^{2}}{(2n)!} = \frac{10\pi\sqrt{3}}{81} + \frac{4}{3} \tag{3.8}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^{2}\,n^{3}}{(2n)!} = \frac{74\pi\sqrt{3}}{243} + \frac{10}{3} \tag{3.9}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^{2}\,n^{4}}{(2n)!} = \frac{238\pi\sqrt{3}}{243} + \frac{32}{3} \tag{3.10}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^{2}\,n^{5}}{(2n)!} = \frac{938\pi\sqrt{3}}{243} + 42 \tag{3.11}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(-1)^n = -\frac{4\sqrt{5}\ln\phi}{25} + \frac{4}{5} \tag{3.12}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(-1)^n\, n = -\frac{4\sqrt{5}\ln\phi}{125} - \frac{6}{25} \tag{3.13}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(-1)^n\, n^2 = \frac{4\sqrt{5}\ln\phi}{125} - \frac{4}{25} \tag{3.14}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(-1)^n\, n^3 = \frac{28\sqrt{5}\ln\phi}{625} + \frac{2}{125} \tag{3.15}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(-1)^n\, n^4 = \frac{4\sqrt{5}\ln\phi}{3125} + \frac{136}{625} \tag{3.16}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}(-1)^n\, n^5 = -\frac{1412\sqrt{5}\ln\phi}{15625} + \frac{742}{3125} \tag{3.17}$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$ is the golden-ratio.

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(-1)^n}{2^n} = -\frac{4\ln 2}{27} + \frac{8}{9} \tag{3.18}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(-1)^n\, n}{2^n} = -\frac{4\ln 2}{81} - \frac{4}{27} \tag{3.19}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(-1)^n\, n^2}{2^n} = \frac{4\ln 2}{729} - \frac{32}{243} \tag{3.20}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(-1)^n\, n^3}{2^n} = \frac{220\ln 2}{6561} - \frac{140}{2187} \tag{3.21}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(-1)^n\, n^4}{2^n} = \frac{76\ln 2}{2187} + \frac{40}{729} \tag{3.22}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(-1)^n\, n^5}{2^n} = -\frac{196\ln 2}{59049} + \frac{3836}{19683} \tag{3.23}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, 2^n = \frac{\pi}{2} + 2 \tag{3.24}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n\, 2^n = \pi + 3 \tag{3.25}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^2\, 2^n = \frac{7\pi}{2} + 11 \tag{3.26}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^3\, 2^n = \frac{35\pi}{2} + 55 \tag{3.27}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^4\, 2^n = 113\pi + 355 \tag{3.28}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^5\, 2^n = \frac{1787\pi}{2} + 2807 \tag{3.29}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^6\, 2^n = \frac{16717\pi}{2} + 26259 \tag{3.30}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^7\, 2^n = 90280\pi + 283623 \tag{3.31}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^8\, 2^n = \frac{2211181\pi}{2} + 3473315 \tag{3.32}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^9\, 2^n = \frac{30273047\pi}{2} + 47552791 \tag{3.33}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^{10}\, 2^n = 229093376\pi + 719718067 \tag{3.34}$$

The last set of examples—particularly Eq. (3.28)—indicate a very interesting, unexpected result. If $m \in \mathbb{N}$, then

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}\, n^m\, 2^n = a_m\pi + b_m \tag{3.35}$$

where $2\,a_m, b_m \in \mathbb{N}$, and $b_m/a_m \to \pi$ as $m \to \infty$. The first ten "convergents" $c_m = b_m/a_m$ are

$$\begin{aligned}
c_0 &= 4 \\
c_1 &= 3 \\
c_2 &= \frac{22}{7} = 3.\overline{142857} \\
c_3 &= \frac{22}{7} = 3.\overline{142857} \\
c_4 &= \frac{355}{133} = 3.14159292035... \\
c_5 &= \frac{5614}{1787} = 3.14157806379... \\
c_6 &= \frac{52518}{16717} = 3.14159239097... \\
c_7 &= \frac{283623}{90280} = 3.14159282233... \\
c_8 &= \frac{6946630}{2211181} = 3.14159266021... \\
c_9 &= \frac{95105582}{30273047} = 3.14159265171... \\
c_{10} &= \frac{719718067}{229093376} = 3.14159265347...
\end{aligned}$$

Empirically, this result implies the irrationality of π (see Niven [9]) since we have infinitely many pairs of integers $(2\,a_m, 2\,b_m)$ such that $0 < |2\,a_m\pi - 2\,b_m| < \epsilon$ for any $\epsilon > 0$. Recently, I have discovered a wonderful paper by Lehmer [6] in which a similar observation is made, along with many interesting evaluations of series involving the central binomial coefficient $\binom{2n}{n}$.

Now consider the series:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{2n+1} &= \sqrt{\frac{\pi}{4x}} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n+1)!} \frac{(4x)^{n+\frac{1}{2}}}{\sqrt{\pi}} = \sqrt{\frac{\pi}{4x}} \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \sum_{n=0}^{\infty} x^n \\
&= \sqrt{\frac{\pi}{4x}} \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} \frac{1}{1-x} = \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}}.
\end{aligned} \tag{3.36}$$

By observing that

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{2n+1} = 1 + \sum_{n=0}^{\infty} \frac{((n-1)!)^2 n^2 (4x)^n}{(2n-2)!(2n-1)2n(2n+1)}, \tag{3.37}$$

we have, by letting $n-1=m$,

$$1+\sum_{m=0}^{\infty}\frac{(m!)^2(m+1)^2(4x)^{m+1}}{(2m)!(2m+1)2(m+1)(2m+3)}=\frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}. \tag{3.38}$$

¿From this, we obtain

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+3}=\left(\frac{2}{x}-1\right)\frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}-\frac{2}{x} \tag{3.39}$$

by partial fraction decomposition. This transformation can be employed successively to obtain the closed form representation of

$$\boxed{\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+2m-1}} \tag{II}$$

for all $m\in\mathbb{N}$. In particular, we find that

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+5}=\left(\frac{8}{3x^2}-\frac{4}{3x}-\frac{1}{3}\right)\frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}-\frac{8}{3x^2}-\frac{4}{9x} \tag{3.40}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+7}=\left(\frac{16}{5x^3}-\frac{8}{5x^2}-\frac{2}{5x}-\frac{1}{5}\right)\frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}-\frac{16}{5x^3}-\frac{8}{15x^2}-\frac{6}{25x} \tag{3.41}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(4x)^n}{2n+9}=\left(\frac{128}{35x^4}-\frac{64}{35x^3}-\frac{16}{35x^2}-\frac{8}{35x}-\frac{1}{7}\right)\frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}-\frac{128}{35x^4}-\frac{64}{105x^3}-\frac{48}{175x^2}-\frac{8}{49x}. \tag{3.42}$$

Examples:

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{2n+1}=\frac{2\pi\sqrt{3}}{9} \tag{3.43}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{2n+3}=\frac{14\pi\sqrt{3}}{9}-8 \tag{3.44}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{2n+5}=\frac{74\pi\sqrt{3}}{9}-\frac{400}{9} \tag{3.45}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{2n+7}=\frac{1774\pi\sqrt{3}}{45}-\frac{16072}{75} \tag{3.46}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{2n+9}=\frac{56758\pi\sqrt{3}}{315}-\frac{3602528}{3675} \tag{3.47}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+1} = \frac{4\sqrt{5}\ln\phi}{5} \tag{3.48}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+3} = -\frac{36\sqrt{5}\ln\phi}{5} + 8 \tag{3.49}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+5} = \frac{572\sqrt{5}\ln\phi}{15} - \frac{368}{9} \tag{3.50}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+7} = -\frac{916\sqrt{5}\ln\phi}{5} + \frac{14792}{75} \tag{3.51}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2n+9} = \frac{29308\sqrt{5}\ln\phi}{35} - \frac{3311008}{3675} \tag{3.52}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+1)} = \frac{4\ln 2}{3} \tag{3.53}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+3)} = -\frac{68\ln 2}{3} + 16 \tag{3.54}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+5)} = \frac{724\ln 2}{3} - \frac{1504}{9} \tag{3.55}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+7)} = -\frac{34756\ln 2}{15} + \frac{120464}{75} \tag{3.56}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+9)} = \frac{2224364\ln 2}{105} - \frac{53963072}{3675} \tag{3.57}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+1} = \frac{\pi}{2} \tag{3.58}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+3} = \frac{3\pi}{2} - 4 \tag{3.59}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+5} = \frac{23\pi}{6} - \frac{104}{9} \tag{3.60}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+7} = \frac{91\pi}{10} - \frac{2116}{75} \tag{3.61}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{2n+9} = \frac{1451\pi}{70} - \frac{238192}{3675} \tag{3.62}$$

Now consider

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}\frac{(4x)^n}{n} = \sqrt{\pi x}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{x^n}{n} = -\sqrt{\pi x}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\ln(1-x)$$

$$= \frac{2\sqrt{x}\arcsin\sqrt{x}}{\sqrt{1-x}}$$

for all $m \in \mathbb{N}$. In particular, we find that

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n+3} = \left(\frac{15}{4x^2} - \frac{5}{4x} - \frac{1}{2} \right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{15 \arcsin^2 \sqrt{x}}{8x^3} - \frac{5}{8x} - \frac{15}{8x^2} \tag{3.68}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n+4} = \left(\frac{35}{8x^3} - \frac{35}{24x^2} - \frac{7}{12x} - \frac{1}{3} \right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{35 \arcsin^2 \sqrt{x}}{16x^4} - \frac{7}{18x} - \frac{35}{48x^2} - \frac{35}{16x^3} \tag{3.69}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n+5} = \left(\frac{315}{64x^4} - \frac{105}{64x^3} - \frac{21}{32x^2} - \frac{3}{8x} - \frac{1}{4} \right) \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}} - \frac{315 \arcsin^2 \sqrt{x}}{128x^5} - \frac{9}{32x} - \frac{7}{16x^2} - \frac{105}{128x^3} - \frac{315}{128x^4}. \tag{3.70}$$

Examples:

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n} = \frac{\pi\sqrt{3}}{9} \tag{3.71}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^{n+1}}{n} = \frac{2\sqrt{5} \ln \phi}{5} \tag{3.72}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^{n+1}}{n\, 2^n} = \frac{\ln 2}{3} \tag{3.73}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n} = \frac{\pi}{2} \tag{3.74}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n^2} = \frac{\pi^2}{18} \tag{3.75}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^{n+1}}{n^2} = 2 \ln^2 \phi \tag{3.76}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^{n+1}}{n^2\, 2^n} = \frac{\ln^2 2}{2} \tag{3.77}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n^2} = \frac{\pi^2}{8} \tag{3.78}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+1} = \frac{4\pi\sqrt{3}}{9} - \frac{\pi^2}{9} \tag{3.79}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+2} = \frac{22\pi\sqrt{3}}{9} - \frac{2\pi^2}{3} - 6 \tag{3.80}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+3} = \frac{109\pi\sqrt{3}}{9} - \frac{10\pi^2}{3} - \frac{65}{2} \tag{3.81}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+4} = \frac{508\pi\sqrt{3}}{9} - \frac{140\pi^2}{9} - \frac{1379}{9} \tag{3.82}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n+5} = \frac{4571\pi\sqrt{3}}{18} - 70\pi^2 - \frac{5525}{8} \tag{3.83}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+1} = \frac{8\sqrt{5}\ln\phi}{5} - 4\ln^2\phi \tag{3.84}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+2} = -\frac{52\sqrt{5}\ln\phi}{5} + 24\ln^2\phi + 6 \tag{3.85}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+3} = \frac{258\sqrt{5}\ln\phi}{5} - 120\ln^2\phi - \frac{55}{2} \tag{3.86}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+4} = -\frac{3616\sqrt{5}\ln\phi}{15} + 560\ln^2\phi + \frac{1169}{9} \tag{3.87}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{n+5} = \frac{5423\sqrt{5}\ln\phi}{5} - 2520\ln^2\phi - \frac{4667}{8} \tag{3.88}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+1)} = \frac{8\ln 2}{3} - 2\ln^2 2 \tag{3.89}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+2)} = -\frac{100\ln 2}{3} + 24\ln^2 2 + 12 \tag{3.90}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+3)} = \frac{998\ln 2}{3} - 240\ln^2 2 - 115 \tag{3.91}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+4)} = -\frac{9316\ln 2}{3} + 2240\ln^2 2 + \frac{9688}{9} \tag{3.92}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+5)} = \frac{83843\ln 2}{3} - 20160\ln^2 2 - \frac{38743}{4} \tag{3.93}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+1} = \pi - \frac{\pi^2}{8} \tag{3.94}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+2} = \frac{5\pi}{2} - \frac{3\pi^2}{8} - 3 \tag{3.95}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+3} = 6\pi - \frac{15\pi^2}{16} - \frac{35}{4} \tag{3.96}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+4} = \frac{83\pi}{6} - \frac{35\pi^2}{16} - \frac{763}{36} \tag{3.97}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n+5} = 31\pi - \frac{315\pi^2}{64} - \frac{193}{4} \tag{3.98}$$

Now by letting $x = 4y^2$ in Eq. (3.36), we obtain

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4y)^{2n}}{2n+1} = \frac{\arcsin 2y}{2y\sqrt{1-4y^2}} \tag{3.99}$$

which, upon integration, implies

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^{2n+1}}{(2n+1)^2} &= 4\int_0^x \frac{\arcsin 2y}{2y\sqrt{1-4y^2}}\, dy \\ &= 2\arcsin 2x \left[\ln\left(1 - e^{i\arcsin 2x}\right) - \ln\left(1 + e^{i\arcsin 2x}\right)\right] \\ &\quad +2i\left[\mathrm{Li}_2\left(-e^{i\arcsin 2x}\right) - \mathrm{Li}_2\left(e^{i\arcsin 2x}\right)\right] + \frac{\pi^2}{2} i \end{aligned} \tag{3.100}$$

By dividing Eq. (3.100) by $4x$ and replacing x by $\frac{\sqrt{x}}{2}$, we have that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(2n+1)^2} &= \frac{\arcsin\sqrt{x}}{\sqrt{x}} \left[\ln\left(1 - e^{i\arcsin\sqrt{x}}\right) - \ln\left(1 + e^{i\arcsin\sqrt{x}}\right)\right] \\ &\quad + \frac{i}{\sqrt{x}}\left[\mathrm{Li}_2\left(-e^{i\arcsin\sqrt{x}}\right) - \mathrm{Li}_2\left(e^{i\arcsin\sqrt{x}}\right)\right] + \frac{\pi^2}{4\sqrt{x}} i \\ &= \frac{\arcsin\sqrt{x}}{\sqrt{x}} \ln\left[\tanh\left(-\frac{1}{2} i \arcsin\sqrt{x}\right)\right] \\ &\quad - \frac{2i}{\sqrt{x}}\, \chi_2\left(e^{i\arcsin\sqrt{x}}\right) + \frac{\pi^2}{4\sqrt{x}} i, \end{aligned} \tag{3.101}$$

where $\chi_2(z) = \frac{1}{2}\mathrm{Li}_2(z) - \frac{1}{2}\mathrm{Li}_2(-z)$ is Legendre's Chi-function. Thus, given that we can explicitly sum any series of the type (II) and carry out an integration by parts similar to

(3.100), this technique can be employed to sum any series of the type

$$\boxed{\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(2n+2m-1)^2}} \tag{IV}$$

for all $m \in \mathbb{N}$. In particular, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(2n+3)^2} &= -\frac{3}{x} + \frac{\sqrt{1-x}}{x^{3/2}} \arcsin\sqrt{x} + \frac{\pi^2}{2x^{3/2}} i \\
&\quad + \frac{2\arcsin\sqrt{x}}{x^{3/2}} \ln\left[\tanh\left(-\frac{1}{2} i \arcsin\sqrt{x}\right)\right] \\
&\quad - \frac{2i}{x^{3/2}} \chi_2\left(e^{i\arcsin\sqrt{x}}\right) \qquad (3.102) \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(2n+5)^2} &= -\frac{5}{27x} - \frac{38}{9x^2} + \left(\frac{14}{9x^{5/2}} + \frac{1}{9x^{3/2}}\right)\sqrt{1-x} \arcsin\sqrt{x} \\
&\quad + \frac{2\pi^2}{3x^{5/2}} i + \frac{8\arcsin\sqrt{x}}{3x^{5/2}} \ln\left[\tanh\left(-\frac{1}{2} i \arcsin\sqrt{x}\right)\right] \\
&\quad - \frac{16i}{3x^{5/2}} \chi_2\left(e^{i\arcsin\sqrt{x}}\right) \qquad (3.103) \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(2n+7)^2} &= -\frac{7}{125x} - \frac{6}{25x^2} - \frac{388}{75x^3} \\
&\quad + \left(\frac{148}{75x^{7/2}} + \frac{14}{75x^{5/2}} + \frac{1}{25x^{3/2}}\right)\sqrt{1-x} \arcsin\sqrt{x} \\
&\quad + \frac{4\pi^2}{5x^{7/2}} i + \frac{16\arcsin\sqrt{x}}{5x^{7/2}} \ln\left[\tanh\left(-\frac{1}{2} i \arcsin\sqrt{x}\right)\right] \\
&\quad - \frac{32i}{5x^{7/2}} \chi_2\left(e^{i\arcsin\sqrt{x}}\right) \qquad (3.104) \\
\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(2n+9)^2} &= -\frac{9}{343x} - \frac{422}{6125x^2} - \frac{1048}{3675x^3} - \frac{21968}{3675x^4} \\
&\quad + \left(\frac{8528}{3675x^{9/2}} + \frac{904}{3675x^{7/2}} + \frac{86}{1225x^{5/2}} + \frac{1}{49x^{3/2}}\right)\sqrt{1-x} \arcsin\sqrt{x} \\
&\quad + \frac{32\pi^2}{35x^{9/2}} i + \frac{128\arcsin\sqrt{x}}{35x^{9/2}} \ln\left[\tanh\left(-\frac{1}{2} i \arcsin\sqrt{x}\right)\right] \\
&\quad - \frac{256i}{35x^{9/2}} \chi_2\left(e^{i\arcsin\sqrt{x}}\right) \qquad (3.105)
\end{aligned}$$

In order to give example summations of (IV), we require the following evaluations:

$$\chi_2(i) = iG \tag{3.106}$$

$$\chi_2\left(\sqrt{2}-1\right) = \frac{\pi^2}{16} - \frac{\ln^2(1+\sqrt{2})}{4} \tag{3.107}$$

$$\chi_2\left(\sqrt{5}-2\right) = \frac{\pi^2}{24} - \frac{3\ln^2\phi}{4} \tag{3.108}$$

$$\chi_2\left(\phi-1\right) = \frac{\pi^2}{12} - \frac{3\ln^2\phi}{4} \tag{3.109}$$

$$\chi_2(1/\sqrt{2}) = \frac{\ln^2 2}{8} - \frac{\pi^2}{48} + \mathrm{Li}_2(1/\sqrt{2}) \tag{3.110}$$

$$\chi_2\left(e^{\pi i/3}\right) = \frac{\pi^2}{24} + \frac{5i\sqrt{3}}{4}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2} - \frac{5\pi^2\sqrt{3}}{54}i \tag{3.111}$$

$$\chi_2\left(e^{\pi i/4}\right) = \frac{\pi^2}{16} + i\sqrt{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^2} - \frac{\pi^2}{16}i \tag{3.112}$$

$$\chi_2\left(e^{\pi i/6}\right) = \frac{\pi^2}{12} + \frac{2i}{3}G \tag{3.113}$$

Here, (3.106) is trivial; (3.107), (3.108), and (3.109) are given in L. Lewin's ***Polylogarithms and Associated Functions*** [8]; (3.111) and (3.112) are easily derived from results found in B. C. Berndt and P. T. Joshi's ***Chapter 9 of Ramanujan's Second Notebook*** [2]; and (3.110) and (3.113) can be obtained by methods illustrated in [8]. Thus, we can now give the following sets of examples, of which the first evaluation of each can be found in [2]:

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{4^n}{(2n+1)^2} = 2\,G \tag{3.114}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{4^n}{(2n+3)^2} = 4\,G - 3 \tag{3.115}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{4^n}{(2n+5)^2} = \frac{16}{3}\,G - \frac{119}{27} \tag{3.116}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{4^n}{(2n+7)^2} = \frac{32}{5}\,G - \frac{2051}{375} \tag{3.117}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{4^n}{(2n+9)^2} = \frac{256}{35}\,G - \frac{272599}{42875} \tag{3.118}$$

By induction,

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{4^n}{(2n+2m+1)^2} = \frac{(m!)^2}{(2m)!}\,4^m\,(2\,G) - q_m, \tag{3.119}$$

where $q_m \in \mathbb{Q}$, $m \in \mathbb{N} \cup \{0\}$, and obviously

$$\lim_{m\to\infty} \frac{(2m)!}{(m!)^2\, 4^m}\, q_m = 2\, G. \tag{3.120}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(2n+1)^2} = \frac{5}{9}\psi'(1/3) - \frac{\pi\sqrt{3}\ln 3}{9} - \frac{10\pi^2}{27} \tag{3.121}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(2n+3)^2} = \frac{40}{27}\psi'(1/3) - \frac{8\pi\sqrt{3}\ln 3}{27} - \frac{80\pi^2}{81} + \frac{4\pi\sqrt{3}}{27} - 4 \tag{3.122}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(2n+5)^2} = \frac{640}{243}\psi'(1/3) - \frac{128\pi\sqrt{3}\ln 3}{243} - \frac{1280\pi^2}{729} + \frac{236\pi\sqrt{3}}{729} - \frac{628}{81} \tag{3.123}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(2n+7)^2} = \frac{1024}{243}\psi'(1/3) - \frac{1024\pi\sqrt{3}\ln 3}{1215} - \frac{2048\pi^2}{729} + \frac{10252\pi\sqrt{3}}{18225} - \frac{129236}{10125} \tag{3.124}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(2n+9)^2} = \frac{32768}{5103}\psi'(1/3) - \frac{32768\pi\sqrt{3}\ln 3}{25515} - \frac{65536\pi^2}{15309} + \frac{2401988\pi\sqrt{3}}{2679075} - \frac{205516468}{10418625} \tag{3.125}$$

where

$$\psi'(1/3) = 9\sum_{k=0}^{\infty} \frac{1}{(3k+1)^2}.$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{(2n+1)^2}=\frac{8}{3}G-\frac{\pi}{3}\ln(2+\sqrt{3}) \tag{3.126}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{(2n+3)^2}=\frac{64}{3}G-\frac{8\pi}{3}\ln(2+\sqrt{3})+\frac{2\pi\sqrt{3}}{3}-12 \tag{3.127}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{(2n+5)^2}=\frac{1024}{9}G-\frac{128\pi}{9}\ln(2+\sqrt{3})+\frac{38\pi\sqrt{3}}{9}-\frac{1844}{27} \tag{3.128}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{(2n+7)^2}=\frac{8192}{15}G-\frac{1024\pi}{15}\ln(2+\sqrt{3})+\frac{1618\pi\sqrt{3}}{75}-\frac{125684}{375} \tag{3.129}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{(2n+7)^2}=\frac{8192}{15}G-\frac{1024\pi}{15}\ln(2+\sqrt{3})+\frac{1618\pi\sqrt{3}}{75}-\frac{125684}{375} \tag{3.130}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{(2n+9)^2}=\frac{262144}{105}G-\frac{32768\pi}{105}\ln(2+\sqrt{3})+\frac{374242\pi\sqrt{3}}{3675}-\frac{66445364}{42875} \tag{3.131}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{2^n}{(2n+1)^2}=4\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^2}-\frac{\pi\sqrt{2}}{4}\ln(1+\sqrt{2})-\frac{\pi^2\sqrt{2}}{8} \tag{3.132}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{2^n}{(2n+3)^2}=16\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^2}-\pi\sqrt{2}\ln(1+\sqrt{2})-\frac{\pi^2\sqrt{2}}{2}+\frac{\pi}{2}-6 \tag{3.133}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{2^n}{(2n+5)^2}=\frac{128}{3}\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^2}-\frac{8\pi\sqrt{2}}{3}\ln(1+\sqrt{2})-\frac{4\pi^2\sqrt{2}}{3}+\frac{29\pi}{18}-\frac{466}{27} \tag{3.134}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{2^n}{(2n+7)^2}=\frac{512}{5}\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^2}-\frac{32\pi\sqrt{2}}{5}\ln(1+\sqrt{2})-\frac{16\pi^2\sqrt{2}}{5}+\frac{623\pi}{150}-\frac{15922}{375} \tag{3.135}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{2^n}{(2n+9)^2}=\frac{8192}{35}\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k+1)^2}-\frac{512\pi\sqrt{2}}{35}\ln(1+\sqrt{2})-\frac{256\pi^2\sqrt{2}}{35}+\frac{72431\pi}{7350}-\frac{12637718}{128625}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n 4^n}{(2n+1)^2} = \frac{\pi^2}{8} - \frac{1}{2} \ln^2(1+\sqrt{2}) \tag{3.136}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n 4^n}{(2n+3)^2} = -\frac{\pi^2}{4} + \ln^2(1+\sqrt{2}) - \sqrt{2}\ln(1+\sqrt{2}) + 3 \tag{3.137}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n 4^n}{(2n+5)^2} = \frac{\pi^2}{3} - \frac{4}{3} \ln^2(1+\sqrt{2}) + \frac{13\sqrt{2}}{9}\ln(1+\sqrt{2}) - \frac{109}{27} \tag{3.138}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n 4^n}{(2n+7)^2} = -\frac{2\pi^2}{5} + \frac{8}{5} \ln^2(1+\sqrt{2}) - \frac{137\sqrt{2}}{75}\ln(1+\sqrt{2}) + \frac{1871}{375} \tag{3.139}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n 4^n}{(2n+9)^2} = \frac{16\pi^2}{35} - \frac{64}{35} \ln^2(1+\sqrt{2}) + \frac{7807\sqrt{2}}{3675}\ln(1+\sqrt{2}) - \frac{737687}{128625} \tag{3.140}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(2n+1)^2} = \frac{\pi^2}{6} - 3\ln^2\phi \tag{3.141}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(2n+3)^2} = -\frac{4\pi^2}{3} + 24\ln^2\phi - 4\sqrt{5}\ln\phi + 12 \tag{3.142}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(2n+5)^2} = \frac{64\pi^2}{9} - 128\ln^2\phi + \frac{220\sqrt{5}}{9}\ln\phi - \frac{1804}{27} \tag{3.143}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(2n+7)^2} = -\frac{512\pi^2}{15} + \frac{3072}{5}\ln^2\phi - \frac{1852\sqrt{5}}{15}\ln\phi + \frac{122804}{375} \tag{3.144}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(2n+9)^2} = \frac{16384\pi^2}{105} - \frac{98304}{35}\ln^2\phi + \frac{425828\sqrt{5}}{735}\ln\phi - \frac{194614052}{128625} \tag{3.145}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+1)^2} = \frac{7\pi^2\sqrt{2}}{12} - \frac{\sqrt{2}}{2} \ln^2 2 - \sqrt{2} \ln 2 \ \ln(3+2\sqrt{2}) - 4\sqrt{2}\ \mathrm{Li}_2(1/\sqrt{2}) \tag{3.146}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+3)^2} = -\frac{20\pi^2\sqrt{2}}{3} + 8\sqrt{2} \ln^2 2 + 24 + 16\sqrt{2} \ln 2 \ \ln(3+2\sqrt{2}) - 12 \ln 2 + 64\sqrt{2}\ \mathrm{Li}_2(1/\sqrt{2}) \tag{3.147}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+5)^2} = \frac{896\pi^2\sqrt{2}}{9} - \frac{256\sqrt{2}}{3} \ln^2 2 - \frac{7256}{27} - \frac{512\sqrt{2}}{3} \ln 2 \ \ln(3+2\sqrt{2}) + 148 \ln 2 - \frac{2048\sqrt{2}}{3}\ \mathrm{Li}_2(1/\sqrt{2}) \tag{3.148}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+7)^2} = -\frac{14336\pi^2\sqrt{2}}{15} + \frac{4096\sqrt{2}}{5} \ln^2 2 + \frac{987688}{375} + \frac{8192\sqrt{2}}{5} \ln 2 \ \ln(3+2\sqrt{2}) - \frac{37452}{25} \ln 2 + \frac{32768\sqrt{2}}{5}\ \mathrm{Li}_2(1/\sqrt{2}) \tag{3.149}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(2n+9)^2} = \frac{131072\pi^2\sqrt{2}}{15} - \frac{262144\sqrt{2}}{35} \ln^2 2 - \frac{1043697496}{42875} - \frac{524288\sqrt{2}}{35} \ln 2 \ \ln(3+2\sqrt{2}) + \frac{17241876}{1225} \ln 2 - \frac{2097152\sqrt{2}}{35}\ \mathrm{Li}_2(1/\sqrt{2}) \tag{3.150}$$

Since there is an inherent difficulty in integrating the appropriate expressions resulting from Eqs (3.100-3.105), we are unable to sum series of the type

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(2n+2m-1)^k} \tag{V}$$

for $k > 2$ and $m \in \mathbb{N}$. However, if we divide Eq. (3.64) by x and integrate, we obtain the closed form representation of

$$\boxed{\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n^3}.} \tag{VI}$$

Namely, we have that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(4x)^n}{n^3} &= \int_0^x \frac{2\arcsin^2\sqrt{y}}{y}dy \\ &= \frac{4i}{3}\arcsin^3\sqrt{x}+4i\arcsin\sqrt{x}\ \mathrm{Li}_2\left(e^{-2i\arcsin\sqrt{x}}\right) \\ &\quad +4\arcsin^2\sqrt{x}\ \ln\left(1-e^{-2i\arcsin\sqrt{x}}\right) \\ &\quad +2\ \mathrm{Li}_3\left(e^{-2i\arcsin\sqrt{x}}\right)-2\ \zeta(3). \qquad (3.151)\end{aligned}$$

Noting that [8]

$$\mathrm{Li}_2(-1) = -\frac{\pi^2}{12} \qquad (3.152)$$

$$\mathrm{Li}_3(-1) = -\frac{3\,\zeta(3)}{4} \qquad (3.153)$$

$$\mathrm{Li}_2(2) = \frac{\pi^2}{4}-i\pi\ln 2 \qquad (3.154)$$

$$\mathrm{Li}_3(2) = \frac{7\,\zeta(3)}{8}+\frac{\pi^2\ln 2}{4}-i\frac{\pi\ln^2 2}{2} \qquad (3.155)$$

$$\mathrm{Li}_2(-i) = -\frac{\pi^2}{48}-iG \qquad (3.156)$$

$$\mathrm{Li}_3(-i) = -\frac{3\,\zeta(3)}{32}-i\frac{\pi^3}{32} \qquad (3.157)$$

$$\mathrm{Li}_2\left(\phi^2\right) = \frac{4\pi^2}{15}-\ln^2\phi-i2\pi\ln\phi \qquad (3.158)$$

$$\mathrm{Li}_3\left(\phi^2\right) = -\frac{2\ln^3\phi}{3}+\frac{8\pi^2\ln\phi}{15}+\frac{4\,\zeta(3)}{5}-i2\pi\ln^2\phi \qquad (3.159)$$

$$\mathrm{Li}_2\left(e^{-\pi i/3}\right) = \frac{\pi^2}{36}-i\frac{3\sqrt{3}}{2}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2}+i\frac{\pi^2\sqrt{3}}{9} \qquad (3.160)$$

$$\mathrm{Li}_3\left(e^{-\pi i/3}\right) = \frac{\zeta(3)}{3}-i\frac{5\pi^3}{162} \qquad (3.161)$$

$$\mathrm{Li}_2\left(e^{-2\pi i/3}\right) = -\frac{\pi^2}{18}-i\sqrt{3}\sum_{k=0}^{\infty}\frac{1}{(3k+1)^2}+i\frac{2\pi^2\sqrt{3}}{27} \qquad (3.162)$$

$$\mathrm{Li}_3\left(e^{-2\pi i/3}\right) = -\frac{4\,\zeta(3)}{9}-i\frac{2\pi^3}{81} \qquad (3.163)$$

we can give the following examples:

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^{n+1}}{n^3} = \frac{2\,\zeta(3)}{5} \tag{3.164}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^{n+1}}{2^n\, n^3} = \frac{\zeta(3)}{4} - \frac{\ln^3 2}{6} \tag{3.165}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n^3} = \frac{\pi\sqrt{3}}{9}\psi'(1/3) - \frac{2\pi^3\sqrt{3}}{27} - \frac{4\,\zeta(3)}{3} \tag{3.166}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n^3} = \frac{\pi^2}{8}\ln 2 - \frac{35\,\zeta(3)}{16} + \pi G \tag{3.167}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{n^3} = \frac{4\pi\sqrt{3}}{27}\psi'(1/3) - \frac{8\pi^3\sqrt{3}}{81} + \frac{2\pi^2}{9}\ln 3 - \frac{26\,\zeta(3)}{9} \tag{3.168}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{4^n}{n^3} = \pi^2 \ln 2 - \frac{7\,\zeta(3)}{2} \tag{3.169}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(3^{n+1}-4)}{n^3} = \frac{2\pi^2}{3}\ln 3 - \frac{10\,\zeta(3)}{3} \tag{3.170}$$

Many of these evaluations have been given independently by several authors unaware of others' work—for instance, this author believed (3.167), (3.170) and (3.171) to be new until he discovered Zucker's paper. In fact, all of these evaluations were obtained by Zucker [14] with the exception of the last evaluation, which is simply obtained by combining (3.168) and (3.170) and eliminating $\psi'(1/3)$.

Paralleling the technique used for (IV), we can sum any series of the type

$$\boxed{\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(n+m)^2}} \tag{VII}$$

for all $m \in \mathbb{N}$. In particular, we find that

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(n+1)^2} = \frac{8 \arcsin^2 \sqrt{x}}{x} - \frac{1}{2x} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n^3} \tag{3.171}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(n+2)^2} &= \frac{\sqrt{1-x}}{x^{3/2}} \arcsin \sqrt{x} + \frac{5 \arcsin^2 \sqrt{x}}{2x^2} - \frac{2}{x} \\ &\quad - \frac{3}{4x^2} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n^3} \end{aligned} \tag{3.172}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(n+3)^2} &= \left(\frac{1}{4x^{3/2}} + \frac{13}{8x^{5/2}} \right) \sqrt{1-x} \arcsin \sqrt{x} + \frac{47 \arcsin^2 \sqrt{x}}{16x^3} \\ &\quad - \frac{3}{8x} - \frac{43}{16x^2} - \frac{15}{16x^3} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n^3} \end{aligned} \tag{3.173}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(n+4)^2} &= \left(\frac{1}{9x^{3/2}} + \frac{31}{72x^{5/2}} + \frac{101}{48x^{7/2}} \right) \sqrt{1-x} \arcsin \sqrt{x} \\ &\quad + \frac{319 \arcsin^2 \sqrt{x}}{96x^4} - \frac{4}{27x} - \frac{17}{36x^2} - \frac{311}{96x^3} \\ &\quad - \frac{35}{32x^4} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n^3} \end{aligned} \tag{3.174}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(n+5)^2} &= \left(\frac{1}{16x^{3/2}} + \frac{19}{96x^{5/2}} + \frac{221}{384x^{7/2}} + \frac{641}{256x^{9/2}} \right) \sqrt{1-x} \arcsin \sqrt{x} \\ &\quad + \frac{1879 \arcsin^2 \sqrt{x}}{512x^5} - \frac{5}{64x} - \frac{103}{576x^2} - \frac{851}{1536x^3} - \frac{1901}{512x^4} \\ &\quad - \frac{315}{256x^5} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{n^3}. \end{aligned} \tag{3.175}$$

Examples:

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+1)^2} = -\frac{4\,\zeta(3)}{5} + 8\ln^2\phi \tag{3.176}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+2)^2} = \frac{24\,\zeta(3)}{5} - 40\ln^2\phi - 4\sqrt{5}\ln\phi + 8 \tag{3.177}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+3)^2} = -24\,\zeta(3) + 188\ln^2\phi - 25\sqrt{5}\ln\phi - \frac{83}{2} \tag{3.178}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+4)^2} = 112\,\zeta(3) - \frac{2552}{3}\ln^2\phi - \frac{1154\sqrt{5}}{9}\ln\phi + \frac{5410}{27} \tag{3.179}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+5)^2} = -504\,\zeta(3) + 3758\ln^2\phi + \frac{7285\sqrt{5}}{12}\ln\phi - \frac{132133}{144} \tag{3.180}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+1)^2} = -\,\zeta(3) + \frac{2}{3}\ln^3 2 + 4\ln^2 2 \tag{3.181}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+2)^2} = 12\,\zeta(3) - 8\ln^3 2 - 40\ln^2 2 - 12\ln 2 + 16 \tag{3.182}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+3)^2} = -120\,\zeta(3) + 80\ln^3 2 + 376\ln^2 2 + 153\ln 2 - 169 \tag{3.183}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+4)^2} = 1120\,\zeta(3) - \frac{2240}{3}\ln^3 2 - \frac{10208}{3}\ln^2 2 - 1576\ln 2 + \frac{44000}{27} \tag{3.184}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{2^n(n+5)^2} = -10080\,\zeta(3) + 6720\ln^3 2 + 30064\ln^2 2 + \frac{59841}{4}\ln 2 - \frac{1075331}{72} \tag{3.185}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+1)^2} = \frac{8\,\zeta(3)}{3} + \frac{4\pi^3\sqrt{3}}{27} + \frac{2\pi^2}{9} - \frac{2\pi\sqrt{3}}{9}\psi'(1/3) \tag{3.186}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+2)^2} = 16\,\zeta(3) + \frac{8\pi^3\sqrt{3}}{9} + \frac{10\pi^2}{9} + \frac{2\pi\sqrt{3}}{3} - \frac{4\pi\sqrt{3}}{3}\psi'(1/3) - 8 \tag{3.187}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+3)^2} = 80\,\zeta(3) + \frac{40\pi^3\sqrt{3}}{9} + \frac{47\pi^2}{9} + \frac{9\pi\sqrt{3}}{2} - \frac{20\pi\sqrt{3}}{3}\psi'(1/3) - \frac{89}{2} \tag{3.188}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+4)^2} = \frac{1120\,\zeta(3)}{3} + \frac{560\pi^3\sqrt{3}}{27} + \frac{638\pi^2}{27} + \frac{71\pi\sqrt{3}}{3} - \frac{280\pi\sqrt{3}}{9}\psi'(1/3) - \frac{5818}{27} \tag{3.189}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+5)^2} = 1680\,\zeta(3) + \frac{280\pi^3\sqrt{3}}{3} + \frac{1879\pi^2}{18} + \frac{2725\pi\sqrt{3}}{24} - 140\pi\sqrt{3}\,\psi'(1/3) - \frac{142435}{144} \tag{3.190}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{(n+1)^2} = \frac{35\,\zeta(3)}{16} - \frac{\pi^2}{8}\ln 2 + \frac{\pi^2}{4} - \pi G \tag{3.191}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{(n+2)^2} = \frac{105\,\zeta(3)}{16} - \frac{3\pi^2}{8}\ln 2 + \frac{5\pi^2}{8} + \frac{\pi}{2} - 3\pi G - 4 \tag{3.192}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{(n+3)^2} = \frac{525\,\zeta(3)}{32} - \frac{15\pi^2}{16}\ln 2 + \frac{47\pi^2}{32} + \frac{7\pi}{4} - \frac{15\pi G}{2} - \frac{23}{2} \tag{3.193}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{(n+4)^2} = \frac{1225\,\zeta(3)}{32} - \frac{35\pi^2}{16}\ln 2 + \frac{319\pi^2}{96} + \frac{169\pi}{36} - \frac{35\pi G}{2} - \frac{3035}{108} \tag{3.194}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{(n+5)^2} = \frac{11025\,\zeta(3)}{128} - \frac{315\pi^2}{64}\ln 2 + \frac{1879\pi^2}{256} + \frac{547\pi}{48} - \frac{315\pi G}{8} - \frac{37273}{576} \tag{3.195}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(n+1)^2} = \frac{52\,\zeta(3)}{27} + \frac{16\pi^3\sqrt{3}}{243} - \frac{4\pi^2}{27}\ln 3 + \frac{8\pi^2}{27} - \frac{8\pi\sqrt{3}}{81}\psi'(1/3) \tag{3.196}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(n+2)^2} = \frac{104\,\zeta(3)}{27} + \frac{32\pi^3\sqrt{3}}{243} - \frac{8\pi^2}{27}\ln 3 + \frac{40\pi^2}{81} + \frac{4\pi\sqrt{3}}{27} - \frac{8}{3} - \frac{16\pi\sqrt{3}}{81}\psi'(1/3) \tag{3.197}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(n+3)^2} = \frac{520\,\zeta(3)}{81} + \frac{160\pi^3\sqrt{3}}{729} - \frac{40\pi^2}{81}\ln 3 + \frac{188\pi^2}{243} + \frac{29\pi\sqrt{3}}{81} - \frac{95}{18} - \frac{80\pi\sqrt{3}}{243}\psi'(1/3) \tag{3.198}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(n+4)^2} = \frac{7280\,\zeta(3)}{729} + \frac{2240\pi^3\sqrt{3}}{6561} - \frac{560\pi^2}{729}\ln 3 + \frac{2552\pi^2}{2187} + \frac{478\pi\sqrt{3}}{729} - \frac{706}{81} - \frac{1120\pi\sqrt{3}}{2187}\psi'(1/3) \tag{3.199}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{3^n}{(n+5)^2} = \frac{3640\,\zeta(3)}{243} + \frac{1120\pi^3\sqrt{3}}{2187} - \frac{280\pi^2}{243}\ln 3 + \frac{3758\pi^2}{2187} + \frac{1049\pi\sqrt{3}}{972} - \frac{5819}{432} - \frac{560\pi\sqrt{3}}{729}\psi'(1/3) \tag{3.200}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{4^n}{(n+1)^2} = \frac{7\,\zeta(3)}{4} - \frac{\pi^2}{2}\ln 2 + \frac{\pi^2}{2} \tag{3.201}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{4^n}{(n+2)^2} = \frac{21\,\zeta(3)}{8} - \frac{2\pi^2}{4}\ln 2 + \frac{5\pi^2}{8} - 2 \tag{3.202}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{4^n}{(n+3)^2} = \frac{105\,\zeta(3)}{32} - \frac{15\pi^2}{16}\ln 2 + \frac{47\pi^2}{64} - \frac{49}{16} \tag{3.203}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{4^n}{(n+4)^2} = \frac{245\,\zeta(3)}{64} - \frac{35\pi^2}{32}\ln 2 + \frac{319\pi^2}{384} - \frac{3335}{864} \tag{3.204}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{4^n}{(n+5)^2} = \frac{2205\,\zeta(3)}{512} - \frac{315\pi^2}{256}\ln 2 + \frac{1879\pi^2}{2048} - \frac{10423}{2304} \tag{3.205}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(3^{n+2}-4)}{(n+1)^2} = \frac{20\,\zeta(3)}{3} - \frac{4\pi^2}{3}\ln 3 + \frac{16\pi^2}{9} \tag{3.206}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(3^{n+3}-4)}{(n+2)^2} = 40\,\zeta(3) - 8\pi^2 \ln 3 + \frac{80\pi^2}{9} + \frac{4\pi\sqrt{3}}{3} - 40 \tag{3.207}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(3^{n+4}-4)}{(n+3)^2} = 200\,\zeta(3) - 40\pi^2 \ln 3 + \frac{376\pi^2}{9} + 11\pi\sqrt{3} - \frac{499}{2} \tag{3.208}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(3^{n+5}-4)}{(n+4)^2} = \frac{2800\,\zeta(3)}{3} - \frac{560\pi^2}{3}\ln 3 + \frac{5104\pi^2}{27} + \frac{194\pi\sqrt{3}}{3} - \frac{33914}{27} \tag{3.209}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(3^{n+6}-4)}{(n+5)^2} = 4200\,\zeta(3) - 840\pi^2 \ln 3 + \frac{7516\pi^2}{9} + \frac{3991\pi\sqrt{3}}{12} - \frac{844277}{144} \tag{3.210}$$

We conclude with noting that the closed form evaluation of

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(4x)^n}{(n+m)^k} \tag{VIII}$$

for all $m \in \mathbb{N}$ and $k' = k \in \mathbb{N}$ involves the evaluations of (VIII) for $k = 0, 1, 2, ..., k'+1$ with $m = 0$. Therefore, we are in general unable to sum this series for $k > 2$ as this would involve repeatedly dividing Eq. (3.153) by x and integrating. However, with $m = 0$ (VIII) has been evaluated for several values of k in the case of $x = \pm\frac{1}{4}$. For example:

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \, 1$$

The first is a classical evaluation due to Comtet [4], and the others (along with evaluations for (VIII) with $m = 0$, $x = \pm\frac{1}{4}$ and $k = 2, 3, ..., 9$–with a few exceptions), are given by Jonathan Borwein, David Broadhurst, and Joel Kamnitzer [3] in terms of "multiple Claussen series" and polylogarithms in the golden ration. In fact, David Bailey and David Broadhurst have obtained the evaluations of (VIII) with $m = 0$ and $x = \frac{1}{4}$ for $k = 2, 3, ..., 20$ using integer relation algorithms [3]. From Eqs. (3.213-2.216), we can obtain the following evaluations:

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+1)^3} = -\frac{16\,\zeta(3)}{3} + \frac{4\pi\sqrt{3}}{9}\psi'(1/3) - \frac{17\pi^4}{1620} - \frac{8\pi^3\sqrt{3}}{27} \tag{3.215}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+2)^3} = -\frac{80\,\zeta(3)}{3} + \frac{20\pi\sqrt{3}}{9}\psi'(1/3) - \frac{17\pi^4}{270} - \frac{40\pi^3\sqrt{3}}{27} + \frac{2\pi^2}{9} + \frac{2\pi\sqrt{3}}{3} - 10 \tag{3.216}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+3)^3} = -\frac{376\,\zeta(3)}{3} + \frac{94\pi\sqrt{3}}{9}\psi'(1/3) - \frac{17\pi^4}{54} - \frac{188\pi^3\sqrt{3}}{27} + \frac{3\pi^2}{2} + \frac{17\pi\sqrt{3}}{4} - \frac{451}{8} \tag{3.217}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+4)^3} = -\frac{5104\,\zeta(3)}{9} + \frac{1276\pi\sqrt{3}}{27}\psi'(1/3) - \frac{119\pi^4}{81} - \frac{2552\pi^3\sqrt{3}}{81} + \frac{641\pi^2}{81} + \frac{391\pi\sqrt{3}}{18} - \frac{29765}{108} \tag{3.218}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+5)^3} = -\frac{7516\,\zeta(3)}{3} + \frac{1879\pi\sqrt{3}}{9}\psi'(1/3) - \frac{119\pi^4}{18} - \frac{3758\pi^3\sqrt{3}}{27} + \frac{8225\pi^2}{216} + \frac{3279\pi\sqrt{3}}{32} - \frac{4396093}{3456} \tag{3.219}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-1)^n}{(n+1)^3}=\frac{13}{3}\ln^4\phi-\frac{14\pi^2}{15}\ln^2\phi+\frac{8\,\zeta(3)}{5}\ln\phi-16\ln\phi\,\mathrm{Li}_3(1/\phi)$$
$$-16\,\mathrm{Li}_4(1/\phi)+\mathrm{Li}_4(1/\phi^2)+\frac{8\,\zeta(3)}{5}+\frac{7\pi^4}{45} \tag{3.220}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-1)^n}{(n+2)^3}=-26\ \ln^4\phi+\frac{28\pi^2}{5}\ln^2\phi-\frac{48\,\zeta(3)}{5}\ln\phi+96\ln\phi\,\mathrm{Li}_3(1/\phi)$$
$$+96\,\mathrm{Li}_4(1/\phi)-6\,\mathrm{Li}_4(1/\phi^2)-8\,\zeta(3)-\frac{14\pi^4}{15}-8\ln^2\phi$$
$$-4\sqrt{5}\ln\phi+10 \tag{3.221}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-1)^n}{(n+3)^3}=130\ \ln^4\phi-28\pi^2\ln^2\phi+48\,\zeta(3)\ln\phi-480\ln\phi\,\mathrm{Li}_3(1/\phi)$$
$$-480\,\mathrm{Li}_4(1/\phi)+30\,\mathrm{Li}_4(1/\phi^2)-\frac{188\,\zeta(3)}{5}-\frac{14\pi^4}{3}$$
$$+54\ln^2\phi+\frac{49\sqrt{5}}{2}\ln\phi-\frac{437}{8} \tag{3.222}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-1)^n}{(n+4)^3}=-\frac{1820}{3}\ \ln^4\phi+\frac{392\pi^2}{3}\ln^2\phi-224\,\zeta(3)\ln\phi$$
$$+2240\ln\phi\,\mathrm{Li}_3(1/\phi)+2240\,\mathrm{Li}_4(1/\phi)-140\,\mathrm{Li}_4(1/\phi^2)$$
$$-\frac{2552\,\zeta(3)}{15}-\frac{196\pi^4}{9}-\frac{2564}{9}\ \ln^2\phi-\frac{3345}{27}\sqrt{5}\ln\phi$$
$$+\frac{28771}{108} \tag{3.223}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\frac{(-1)^n}{(n+5)^3}=2730\ \ln^4\phi-588\pi^2\ln^2\phi+1008\,\zeta(3)\ln\phi$$
$$-10080\ln\phi\,\mathrm{Li}_3(1/\phi)-10080\,\mathrm{Li}_4(1/\phi)+630\,\mathrm{Li}_4(1/\phi^2)$$
$$-\frac{3758\,\zeta(3)}{5}-98\pi^4+\frac{8225}{6}\ln^2\phi+\frac{83575\sqrt{5}}{144}\ln\phi$$
$$-\frac{4243315}{3456} \tag{3.224}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+1)^4} = -\frac{2\pi^2}{9}\,\zeta(3) + \frac{38\,\zeta(5)}{3} - \frac{\pi\sqrt{3}}{108}\,\psi^{(3)}(1/3) + \frac{2\pi^5\sqrt{3}}{81} + \frac{17\pi^4}{810} \tag{3.225}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+2)^4} = &-\frac{4\pi^2}{3}\,\zeta(3) - \frac{16\,\zeta(3)}{3} + \frac{4\pi\sqrt{3}}{9}\,\psi'(1/3) + 76\,\zeta(5) \\ &- \frac{\pi\sqrt{3}}{18}\,\psi^{(3)}(1/3) + \frac{4\pi^5\sqrt{3}}{27} + \frac{17\pi^4}{162} - \frac{8\pi^3\sqrt{3}}{27} \\ &+ \frac{2\pi^2}{9} + \frac{2\pi\sqrt{3}}{3} - 12\end{aligned} \tag{3.226}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+3)^4} = &-\frac{20\pi^2}{3}\,\zeta(3) - 36\,\zeta(3) + 3\pi\sqrt{3}\,\psi'(1/3) + 380\,\zeta(5) \\ &- \frac{5\pi\sqrt{3}}{18}\,\psi^{(3)}(1/3) + \frac{20\pi^5\sqrt{3}}{27} + \frac{799\pi^4}{1620} - 2\pi^3\sqrt{3} \\ &+ \frac{17\pi^2}{12} + \frac{33\pi\sqrt{3}}{8} - \frac{273}{4}\end{aligned} \tag{3.227}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+4)^4} = &-\frac{280\pi^2}{9}\,\zeta(3) - \frac{5128\,\zeta(3)}{27} + \frac{1282\pi\sqrt{3}}{81}\,\psi'(1/3) + \frac{5320\,\zeta(5)}{3} \\ &- \frac{35\pi\sqrt{3}}{27}\,\psi^{(3)}(1/3) + \frac{280\pi^5\sqrt{3}}{81} + \frac{5423\pi^4}{2430} - \frac{2564\pi^3\sqrt{3}}{243} \\ &+ \frac{3523\pi^2}{486} + \frac{2249\pi\sqrt{3}}{108} - \frac{325823}{972}\end{aligned} \tag{3.228}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{(n+5)^4} = &-140\pi^2\,\zeta(3) - \frac{8225\,\zeta(3)}{9} + \frac{8225\pi\sqrt{3}}{108}\,\psi'(1/3) + 7980\,\zeta(5) \\ &- \frac{35\pi\sqrt{3}}{6}\,\psi^{(3)}(1/3) + \frac{140\pi^5\sqrt{3}}{9} + \frac{31943\pi^4}{3240} - \frac{8225\pi^3\sqrt{3}}{162} \\ &+ \frac{88715\pi^2}{2592} + \frac{112439\pi\sqrt{3}}{1152} - \frac{2679859}{1728}\end{aligned} \tag{3.229}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+1)^4} = -\frac{26}{3} \ln^4 \phi + \frac{28\pi^2}{15} \ln^2 \phi - \frac{16\,\zeta(3)}{5} \ln \phi + 32 \ln \phi \operatorname{Li}_3(1/\phi)$$
$$+ 32 \operatorname{Li}_4(1/\phi) + \operatorname{Li}_4(1/\phi^2) - \frac{14\pi^4}{45} - 5 \operatorname{Li}_5(1/\phi^2)$$
$$- 10 \ln \phi \operatorname{Li}_4(1/\phi^2) - 8\,\zeta(3) \ln^2 \phi + \frac{8\pi^2}{9} \ln^3 \phi$$
$$- \frac{8}{3} \ln^5 \phi + 4\,\zeta(5) \tag{3.230}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+2)^4} = \frac{130}{3} \ln^4 \phi - \frac{28\pi^2}{3} \ln^2 \phi - 8 \ln^2 \phi + 16\,\zeta(3) \ln \phi$$
$$- 160 \ln \phi \operatorname{Li}_3(1/\phi) - 4\sqrt{5} \ln \phi - \frac{8\,\zeta(3)}{5} - 160 \operatorname{Li}_4(1/\phi)$$
$$+ 10 \operatorname{Li}_4(1/\phi^2) + \frac{14\pi^4}{9} + 30 \operatorname{Li}_5(1/\phi^2) + 60 \ln \phi \operatorname{Li}_4(1/\phi^2)$$
$$+ 48\,\zeta(3) \ln^2 \phi - \frac{16\pi^2}{3} \ln^3 \phi + 16 \ln^5 \phi - 24\,\zeta(5) + 12 \tag{3.231}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+3)^4} = -\frac{611}{3} \ln^4 \phi + \frac{658\pi^2}{15} \ln^2 \phi + 51 \ln^2 \phi - \frac{376\,\zeta(3)}{5} \ln \phi$$
$$+ 752 \ln \phi \operatorname{Li}_3(1/\phi) + \frac{97\sqrt{5}}{4} \ln \phi + \frac{54\,\zeta(3)}{5} + 752 \operatorname{Li}_4(1/\phi)$$
$$- 47 \operatorname{Li}_4(1/\phi^2) - \frac{329\pi^4}{45} - 150 \operatorname{Li}_5(1/\phi^2) - 300 \ln \phi \operatorname{Li}_4(1/\phi^2)$$
$$+ 240\,\zeta(3) \ln^2 \phi + \frac{80\pi^2}{3} \ln^3 \phi - 80 \ln^5 \phi + 120\,\zeta(5) - \frac{269}{4} \tag{3.232}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+4)^4} = \frac{8294}{9} \ln^4 \phi - \frac{8932\pi^2}{45} \ln^2 \phi - \frac{7046}{27} \ln^2 \phi + \frac{5104\,\zeta(3)}{15} \ln \phi$$
$$- \frac{10208}{3} \ln \phi \operatorname{Li}_3(1/\phi) - \frac{19723\sqrt{5}}{162} \ln \phi - \frac{2564\,\zeta(3)}{45}$$
$$- \frac{10208}{3} \operatorname{Li}_4(1/\phi) + \frac{638}{3} \operatorname{Li}_4(1/\phi^2) + \frac{4466\pi^4}{135}$$
$$+ 700 \operatorname{Li}_5(1/\phi^2) + 1400 \ln \phi \operatorname{Li}_4(1/\phi^2) + 1120\,\zeta(3) \ln^2 \phi$$
$$- \frac{1120\pi^2}{9} \ln^3 \phi + \frac{1120}{3} \ln^5 \phi - 560\,\zeta(5) + \frac{320573}{972} \tag{3.233}$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \frac{(-1)^n}{(n+5)^4} = -\frac{24427}{6} \ln^4 \phi + \frac{13153\pi^2}{15} \ln^2 \phi + \frac{88715}{72} \ln^2 \phi - \frac{7516\,\zeta(3)}{5} \ln \phi$$
$$+ 15032 \ln \phi \,\mathrm{Li}_3(1/\phi) + \frac{983605\sqrt{5}}{1728} \ln \phi + \frac{1645\,\zeta(3)}{6}$$
$$+ 15032\, \mathrm{Li}_4(1/\phi) - \frac{1879}{2} \mathrm{Li}_4(1/\phi^2) - \frac{13153\pi^4}{90}$$
$$- 3150\, \mathrm{Li}_5(1/\phi^2) - 6300 \ln \phi \,\mathrm{Li}_4(1/\phi^2) - 5040\,\zeta(3) \ln^2 \phi$$
$$+ 560\pi^2 \ln^3 \phi - 1680 \ln^5 \phi + 2520\,\zeta(5) - \frac{2634659}{1728} \tag{3.234}$$

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