EXERCISES FOR p-ADIC HODGE THEORY

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1. Introduction

1. Let E be an elliptic curve over $\mathbb{Q}$, defined by an equation

$$y^2 = x^3 + ax + b \quad \text{with } a,\ b \in \mathbb{Q} \text{ and } 4a^3 + 27b^2 \neq 0.$$

(1) Show that every nonvertical line and E have three intersection points, counted with multiplicity.

(2) The group law on E, written additively, is given by the following properties:
  (i) The identity element O is the point at infinity.
  (ii) Given a point P on E, the vertical line passing through it and E have the second intersection point at $-P$.
  (iii) Given two points $P,\ Q$ on E with distinct x-coordinates, the line passing through them and E have the third intersection point at $-(P+Q)$.
  (iv) Given a point P on E, the tangent line to E at P and E have the third intersection point at $-(P+P)$.

Given two arbitrary points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ on E, derive a formula for their sum $P + Q$.

Remark. It is not obvious to verify that the group law on E defined above is indeed associative. For curious readers who attempt to verify this by themselves, there are two possible approaches as follows:

(a) One can use the formula for the group law obtained here for a direct verification.
(b) One can use Riemann-Roch theorem to show that the group law on E agrees with the group law on $\mathrm{Pic}^0(E)$, the degree 0 part of the Picard group $\mathrm{Pic}(E)$.

2. Let E be an elliptic curve over $\mathbb{Q}$ and n be a positive integer.

(1) Show that $E[n](\overline{\mathbb{Q}})$ has n^2 elements.

Hint. Look at the degree of polynomials defining the multiplication by n.

(2) Establish an identification $E[n](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$.

Hint. Apply the fundamental theorem for finitely generated abelian groups after observing that $E[d](\overline{\mathbb{Q}})$ has d^2 elements for each divisor d of n.

Remark. If we replace the base field $\mathbb{Q}$ with another field, the conclusions of this exercise remains valid as long as n is invertible in the base field.

3. Let E be an elliptic curve over $\mathbb{Q}$ and ℓ be a prime number.

(1) Show that the ℓ-adic Tate-module $T_\ell(E)$ is a free $\mathbb{Z}_\ell$-module of rank 2.

(2) Show that $T_\ell(E)$ carries a natural action of $\Gamma_{\mathbb{Q}}$.

Hint. Show first that $E[\ell^v]$ carries a natural action of $\Gamma_{\mathbb{Q}}$ for each positive integer v.

4. In this exercise, we give a simple analogy between the complex conjugation and the p-adic cyclotomic character.

(1) Let μ_∞ denote the group of roots of unity in $\mathbb{C}$. Show that the complex conjugation naturally induces a character
$$\tilde{\chi} : \Gamma_{\mathbb{R}} \longrightarrow \mathrm{Aut}(\mathbb{R}) \cong \mathbb{R}^\times$$
with $\gamma(\zeta) = \zeta^{\tilde{\chi}(\gamma)}$ for every $\gamma \in \Gamma_{\mathbb{R}}$ and $\zeta \in \mu_\infty$.

(2) Let μ_{p^∞} denote the group of p-power roots of unity in $\overline{\mathbb{Q}}_p$. Show that the p-adic cyclotomic character χ yields the relation $\gamma(\zeta) = \zeta^{\chi(\gamma)}$ for every $\gamma \in \Gamma_{\mathbb{Q}_p}$ and $\zeta \in \mu_{p^\infty}$.

5. This exercise requires some knowledge on the étale cohomology and the Hodge theory.

(1) Directly verify the Hodge-Tate decomposition theorem for $\mathbb{P}^1$.

(2) Show that the p-adic de Rham comparison theorem fails if we replace B_{dR} by $\mathbb{C}_p$.

6. Deduce the identification (1.9) from Theorem 1.2.4 and Theorem 2.1.1.

7. Let ν_∞ denote the valuations on B_{dR} and $\mathbb{C}((z^{-1}))$.

(1) Show the identity $\deg(f) = -\nu_\infty(f)$ for every $f \in \mathbb{C}(z)$.

(2) Define the *degree* of each $f \in B_{\mathrm{dR}}$ to be $\deg(f) := -\nu_\infty(f)$. Prove the identity
$$\deg(fg) = \deg(f) + \deg(f) \qquad \text{for any } f,\ g \in B_{\mathrm{dR}}.$$

8. In this exercise, we provide a precise description of the Fargues-Fontaine curve X as a scheme that glues $\mathrm{Spec}\,(B_e)$ and $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ along $\mathrm{Spec}\,(B_{\mathrm{dR}})$; in other words, we prove that the topological space obtained by gluing $\mathrm{Spec}\,(B_e)$ and $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ along $\mathrm{Spec}\,(B_{\mathrm{dR}})$ is naturally a scheme. We define the degree function on B_{dR} as in Exercise 7.

(1) Under the identification $\mathbb{A}^1_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} - \infty$, prove that $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$ is given by
$$\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(U) = \begin{cases} \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(U) & \text{for any open } U \subseteq \mathbb{P}^1_{\mathbb{C}} \text{ with } \infty \notin U, \\ \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(U-\infty)^- & \text{for any open } U \subseteq \mathbb{P}^1_{\mathbb{C}} \text{ with } \infty \in U \end{cases}$$
where we set $\mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(U-\infty)^- := \left\{ f \in \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(U-\infty) : \deg(f) \leq 0 \right\}$.

(2) Let us set $X^\circ := \mathrm{Spec}\,(B_e)$ and denote by ∞ the special point of $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$. Prove that X is indeed a scheme with the structure sheaf given by
$$\mathcal{O}_X(U) = \begin{cases} \mathcal{O}_{X^\circ}(U) & \text{for any open } U \subseteq X \text{ with } \infty \notin U, \\ \mathcal{O}_{X^\circ}(U-\infty)^- & \text{for any open } U \subseteq X \text{ with } \infty \in U \end{cases}$$
where we set $\mathcal{O}_X(U-\infty)^- := \{ f \in \mathcal{O}_{X^\circ}(U-\infty) : \deg(f) \leq 0 \}$.

9. Deduce properties (i), (ii) and (iv) in Theorem 2.1.2 from the original construction of the Fargues-Fontaine curve X, given by gluing $\mathrm{Spec}\,(B_e)$ and $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ along $\mathrm{Spec}\,(B_{\mathrm{dR}})$, and the fact that B_e is a principal ideal domain.

2. Foundations of p-adic Hodge theory

1. For affine group schemes introduced in Example 1.1.8, verify the descriptions of their comultiplication, counit, and coinverse.

2. In this exercise, we study homomorphisms between the group schemes $\mathbb{G}_a$ and $\mathbb{G}_m$.

(1) Show that every homomorphism from $\mathbb{G}_m$ to $\mathbb{G}_a$ is trivial.

(2) If R is reduced, show that every homomorphism from $\mathbb{G}_a$ to $\mathbb{G}_m$ is trivial.

(3) If R contains a nonzero element α with $\alpha^2 = 0$, construct a nonzero homomorphism from $\mathbb{G}_a$ to $\mathbb{G}_m$.

3. Assume that $R = k$ is a field of characteristic p.

(1) Show that the k-algebra homomorphism $k[t] \to k[t]$ which sends t to $t^p - t$ induces a k-group homomorphism $f : \mathbb{G}_a \to \mathbb{G}_a$.

(2) Show that $\ker(f)$ is isomorphic to $\underline{\mathbb{Z}/p\mathbb{Z}}$.

4. Prove that an R-group is separated if and only if its unit section is a closed embedding.

Hint. One can identify the unit section as a base change of the diagonal morphism and conversely identify the diagonal morphism as a base change of the unit section.

5. Assume that $R = k$ is a field of characteristic p.

(1) Verify that the k-group $\alpha_{p^2} := \mathrm{Spec}\,(k[t]/t^{p^2})$ with the natural additive group structure on $\alpha_{p^2}(B) = \left\{ b \in B : b^{p^2} = 0 \right\}$ for each k-algebra B is finite flat of order p^2.

(2) Show that $\alpha_{p^2}^\vee$ admits an isomorphism $\alpha_{p^2}^\vee \cong \mathrm{Spec}\,(k[t,u]/(t^p, u^p))$ with the multiplication on $\alpha_{p^2}^\vee(B) \cong \{ (b_1, b_2) \in B^2 : b_1^p = b_2^p = 0 \}$ for each k-algebra B given by

$$(b_1, b_2) \cdot (b_1', b_2') = \big(b_1 + b_1', b_2 + b_2' - W_1(b_1, b_2)\big)$$

where we write $W_1(t, u) := ((t+u)^p - t^p - u^p)/p \in \mathbb{Z}[t, u]$.

Hint. One can show that a B-algebra homomorphism $B[t, t^{-1}] \to B[t]/(t^{p^2})$ induces a B-group homomorphism $\alpha_{p^2} \to \mathbb{G}_m$ if and only if the image of t is of the form $f(t) = E(b_1 t)E(b_2 t^p)$ with $b_1^p = b_2^p = 0$, where we write $E(t) := \displaystyle\sum_{i=0}^{p-1} \frac{t^i}{i!}$.

(3) For $k = \overline{\mathbb{F}}_p$, show that α_{p^2} fits into a nonsplit short exact sequence

$$\underline{0} \longrightarrow \alpha_p \longrightarrow \alpha_{p^2} \longrightarrow \alpha_p \longrightarrow \underline{0}.$$

6. Assume that $R = k$ is a perfect field.

(1) Given a finite abelian group M with a continuous Γ_k-action, show that the scheme $\underline{M}^{\Gamma_k} := \text{Spec}\,(A)$ for $A := \Big(\prod_{i \in M} \overline{k}\Big)^{\Gamma_k}$ is naturally a finite étale k-group.

Hint. Since M is finite, the Γ_k-action should factor through a finite quotient.

(2) Prove that the inverse functor for the equivalence in Proposition 1.3.4 maps each finite abelian group M with a continuous Γ_k-action to $\underline{M}^{\Gamma_k}$.

(3) Prove that a finite étale group scheme G over a field k is a constant group scheme if and only if the Γ_k-action on $G(\overline{k})$ is trivial.

7. In this exercise, we follow the notes of Pink [Pin, §15] to present a counterexample for Proposition 1.4.15 when k is not perfect. Let us choose $c \in k$ which is not a p-th power and set $G := \coprod_{i=0}^{p-1} G_i$ with $G_i := \text{Spec}\,\big(k[t]/(t^p - c^i)\big)$.

(1) Given a k-algebra B, verify a natural identification

$$G_i(B) \cong \big\{\, b \in B : b^p = c^i \,\big\} \qquad \text{for each } i = 0, \cdots, p-1$$

and show that $G(B)$ is a group with multiplication given by the following maps:

- $m_{ij} : G_i(B) \times G_j(B) \to G_{i+j}(B)$ for $i + j < p$ which sends each (g, g') to gg',
- $m_{ij} : G_i(B) \times G_j(B) \to G_{i+j-p}(B)$ for $i + j \geq p$ which sends each (g, g') to gg'/c.

(2) Show that G yields a nonsplit connected-étale sequence

$$\underline{0} \longrightarrow \mu_p \longrightarrow G \longrightarrow \underline{\mathbb{Z}/p\mathbb{Z}} \longrightarrow \underline{0}.$$

Hint. To show that the sequence does not split, compare G_0 with G_i for $i \neq 0$.

8. Assume that $R = k$ is a field.

(1) If k has characteristic 0, establish a natural identification $\text{End}_{k\text{-grp}}(\mathbb{G}_a) \cong k$.

(2) If k has characteristic p, show that $\text{End}_{k\text{-grp}}(\mathbb{G}_a)$ is isomorphic to the (possibly noncommutative) polynomial ring $k\langle\varphi\rangle$ with $\varphi c = c^p \varphi$ for any $c \in k$.

9. Assume that $R = k$ is a field.

(1) Give a proof of Theorem 1.3.10 when $R = k$ is a field without using Theorem 1.1.16.

Hint. If k has characteristic 0, we can adjust the proof of Proposition 1.5.19 to obtain an isomorphism $G^\circ \simeq \text{Spec}\,(k[t_1, \cdots, t_d])$ for some integer $d \geq 0$ and in turn find $d = 0$ by the fact that G° is finite flat.

(2) Prove Theorem 1.1.16 when $R = k$ is a field.

Hint. If k has characteristic 0, we can deduce the assertion from the corresponding theorem for finite groups by observing that G is étale. If k has characteristic p, we can reduce to the case where G is simple with k algebraically closed.

10. Use the self-duality of elliptic curves to prove that every elliptic curve over $\overline{\mathbb{F}}_p$ is either ordinary or supersingular.

11. Assume that $R = k$ is a perfect field.

(1) Show that the dual of every étale p-divisible group over k is connected.

(2) Show that every p-divisible G over k admits a natural decomposition
$$G \cong G^{\text{ll}} \times G^{\text{mult}} \times G^{\text{ét}}$$
with the following properties:
 (i) G^{ll} is connected with $(G^{\text{ll}})^\vee$ connected.
 (ii) G^{mult} is connected with $(G^{\text{mult}})^\vee$ étale.
 (iii) $G^{\text{ét}}$ is étale with $(G^{\text{ét}})^\vee$ connected.

12. Assume that $R = k$ is a field of characteristic 0. Establish an isomorphism between the formal group laws $\mu_{\widehat{\mathbb{G}_a}}$ and $\mu_{\widehat{\mathbb{G}_m}}$ over k defined as in Example 2.2.3.

Hint. Consider the map $k[[t]] \to k[[t]]$ sending t to $\exp(t) - 1 = \displaystyle\sum_{n=1}^{\infty} \frac{t^n}{n!}$.

13. Let K be a finite extension of $\mathbb{Q}_p$ with uniformizer π and residue field $\mathbb{F}_q$.

(1) Show that there exists a unique formal group law μ_π over $\mathcal{O}_K$ of dimension 1 with an endomorphism $[\pi] : \mathcal{O}_K[[t]] \to \mathcal{O}_K[[t]]$ sending t to $\pi t + t^q$.

(2) Show that μ_π is p-divisible.

Remark. The formal group law μ_π is a *Lubin-Tate formal group law*, introduced by the work of Lubin-Tate [LT65] as a means to construct the totally ramified abelian extensions of K.

14. For a supersingular elliptic curve E over $\overline{\mathbb{F}}_p$, show that $\ker(\varphi_{E[p]})$ is isomorphic to α_p.

15. Recall that every $\alpha \in \mathbb{Z}_p$ admits a unique p-adic expansion $\alpha = \displaystyle\sum_{n=0}^{\infty} a_n p^n$ where each a_n is an integer with $0 \leq a_n < p$.

(1) Show that the 2-adic expansion agrees with the Teichmüler expansion on $\mathbb{Z}_2$.

(2) Show that the p-adic expansion does not agree with the Teichmüler expansion on $\mathbb{Z}_p$ for $p > 2$.

(3) Find the 3-adic expansion for $[2] \in \mathbb{Z}_3$.

(4) Find the first four coefficients of the 5-adic expansion for $[2] \in \mathbb{Z}_5$.

Hint. The Teichmüler lift of an element $a \in \mathbb{F}_p$ is the unique lift $[a] \in \mathbb{Z}_p$ with $[a]^p = [a]$. One can inductively find its image in $\mathbb{Z}_p/p^n\mathbb{Z}_p = \mathbb{Z}/p^n\mathbb{Z}$ for each $n \geq 1$ by Hensel's lemma.

16. Assume that $R = k$ is a perfect field of characteristic p. For each $\lambda \in \mathbb{Q}$, show that there exists a natural isomorphism $N(\lambda)^\vee \cong N(-\lambda)$.

17. Let A be an abelian variety over $\overline{\mathbb{F}}_p$ of dimension g.

(1) Show that the isocrystal $\mathbb{D}(A[p^\infty])[1/p]$ is self-dual by using the fact that A is isogenous to its dual.

(2) If A is *ordinary* in the sense that $A[p](\overline{\mathbb{F}}_p)$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus g}$, show that there exists an isomorphism

$$A[p^\infty] \simeq (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g \times (\mu_{p^\infty})^g.$$

Hint. Show that $A[p^\infty]^\circ$ has étale dual, possibly by establishing an isomorphism $\mathbb{D}(A[p^\infty])[1/p] \simeq N(0)^{\oplus g} \oplus N(1)^{\oplus g}$.

18. Let K be a p-adic field.

(1) Prove that its algebraic closure $\overline{K}$ is not p-adically complete.

Hint. There are at least two ways to proceed as follows:

(a) One can observe that $\overline{K}$ is a countable union of nowhere dense subspaces and apply the Baire category theorem to conclude.

(b) Alternatively, one can use Krasner's lemma to produce a Cauchy sequence in $\overline{K}$ whose limit is not algebraic over K.

(2) Prove that $\mathbb{C}_K$ is not discretely valued.

19. Give a proof of Proposition 3.3.10 for $G = \mu_{p^\infty}$.

20. Let K be a p-adic field and E be an elliptic curve over $\mathcal{O}_K$.

(1) Prove that E gives rise to a Γ_K-equivariant $\mathbb{Z}_p$-linear perfect pairing

$$T_p(E[p^\infty]) \times T_p(E[p^\infty]) \to \mathbb{Z}_p(1). \tag{2.1}$$

(2) Deduce that the determinant character of the Γ_K-representation $T_p(E[p^\infty])$ coincides with the p-adic cyclotomic character.

Remark. The perfect pairing (2.1) coincides with the *Weil pairing* on E.

21. Describe the canonical identification

$$\mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(\mathbb{C}_K(-1), \mathbb{C}_K) \cong H^1(\Gamma_K, \mathbb{C}_K(1))$$

used in the proof of Theorem 3.4.13.

Hint. Given a Γ_K-representation V over $\mathbb{C}_K$ with a Γ_K-equivariant short exact sequence

$$0 \longrightarrow \mathbb{C}_K \longrightarrow V \longrightarrow \mathbb{C}_K(-1) \longrightarrow 0,$$

the action of Γ_K on $V(1)$ admits a matrix representation

$$\begin{pmatrix} \chi & c \\ 0 & 1 \end{pmatrix}$$

for some map $c : \Gamma_K \to \mathbb{C}_K(1)$. Show that c is a 1-cocycle on Γ_K in $\mathbb{C}_K(1)$ with its class in $H^1(\Gamma_K, \mathbb{C}_K(1))$ uniquely determined by the isomorphism class of V.

3. Period rings and functors

1. Let B be a $(\mathbb{Q}_p, \Gamma_K)$ ring.

(1) Show that there exists a natural bijection between $H^1(\Gamma_K, \mathrm{GL}_d(B))$ and the set of equivalence classes of free B-module of rank d with a continuous Γ_K-action.

(2) Show that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ with $d = \dim_{\mathbb{Q}_p}(V)$ is B-admissibile if and only if the Γ_K-action on $V \otimes_{\mathbb{Q}_p} B$ is trivial.

2. Verify that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is $\overline{K}$-admissible if and only if the Γ_K-action on V factors through a finite quotient, as stated in Example 1.1.4.

Hint. Use (a strong version of) Hilbert's Theorem 90 to prove the identity $H^1(\Gamma_K, \mathrm{GL}_d(\overline{K})) = 0$ and apply the previous exercise.

3. Show that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is $\mathbb{C}_K$-admissibile if and only if it is Hodge-Tate with 0 as the unique Hodge-Tate weight.

4. Given an elliptic curve E over $\mathcal{O}_K$, prove that the Γ_K-representation $V_p(E[p^\infty])$ is never unramified.

5. Prove that a p-divisible group G over $\mathcal{O}_K$ is étale if and only if 0 is not a Hodge-Tate weight of $V_p(G)$.

6. Given an abelian variety A over K of dimension g with good reduction, find the multiplicity for each Hodge-Tate weight of the étale cohomology group $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$.

7. Show that B_{dR}^+ is not $(\mathbb{Q}_p, \Gamma_K)$-regular.

8. Show that the category Fil_K is not abelian.

9. Show the (enhanced) functors D_{HT} and D_{dR} are not fully faithful respectively on the categories of Hodge-Tate representations and de Rham representations.

4. The Fargues-Fontaine curve

1. In this exercise, we follow an argument of Fontaine to deduce Corollary 4.1.16 from the following result:

Proposition 4.0.1 (Berger [Ber08])**.** The ring B_e is Bézout; in other words, the sum of two principal ideals in B_e is principal.

Let us define the degree of an element $x \in B_e$ to be the smallest integer d with $x \in t^{-d} B_{\mathrm{dR}}^+$.

(1) Show that $x \in B_e$ is a unit if and only if its degree is 0.

(2) Show that every ideal I of B_e is generated by an arbitrary element of minimial degree.

2. See what happens if we mimic the construction of $\mathcal{O}_h(d, r) := (\pi_{rh,h})_* \mathcal{O}_{rh}(d)$ for $\mathbb{P}^1_k$ with k an arbitrary field.

References

[Ber08] Laurent Berger, *Construction of (ϕ, γ)-modules: p-adic representations and B-pairs*, Algebra & Number Theory **2** (2008), no. 1, 91–120.

[LT65] Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Annals of Math. **81** (1965), no. 2, 380–387.

[Pin] Richard Pink, *Finite group schemes*, `ftp://ftp.math.ethz.ch/users/pink/FGS/CompleteNotes.pdf`.