Decomposition theorem Let Σ = finite or countably [cut off], and consider a Markov chain on Σ with stationary transitions.

Then $\Sigma = \Sigma_T \cup \Sigma_R$ where if $\Sigma_R \neq \emptyset$, $\Sigma_R = \bigcup_\ell C_\ell$, $\{C_\ell\}$ are (non-communicating) closed irreducible sets of recurrent states.

pf: Clearly $\Sigma = \Sigma_T \cup \Sigma_R$ as any state is either transient or recurrent. When $\Sigma_R \neq \emptyset$, let $x \in \Sigma_R$. Let $C_x = \{y : x \to y\}$. Since x is recurrent, $x \to x$ and so $x \in C_x$. Also, as all states which x leads to are recurrent, we have $C_x \subset \Sigma_R$.

Then, we have $\Sigma_R = \bigcup_{x \in \Sigma_R} C_x$.

To finish the proof, we need to show (1) C_x is irreducible, and closed. (2) For $x, y \in \Sigma_R$ either $C_x = C_y$ or $C_x \cap C_y = \emptyset$.

Then, Σ_R is seen as the union of disjoint closed irreducible sets.

(1) irreducible let $y, z \in C_x$. As x recurrent and $x \to y$, we have $y \to x$. But then $y \to x$, $x \to z$ and so from exercise $y \to z$.

closed Suppose $y \in C_x$ and $y \to z$. Then as $x \to y$, we have $x \to z$ $\Rightarrow z \in C_x$ $\Rightarrow C_x$ closed.

(2) If $C_x \cap C_y \neq \emptyset$, let $z \in C_x \cap C_y$. Let also $w \in C_x$. Then $z \to w$ as C_x irreducible. But C_y is closed, so as $z \in C_y$, we have $w \in C_y$. $\Rightarrow$ Every state $w \in C_x$ also belongs to C_y or $C_x \subset C_y$. Similarly $C_y \subset C_x$

$\Rightarrow C_x = C_y$ //.