Midterm Math 468/568 Spring 2016 Name: Solutions

Show all work. Calculators may be used, but all answers must be derived.

1. Consider the Markov chain on $\{1, 2, 3, 4, 5\}$ with transition matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & .2 & 0 & .8 & 0 \\ .1 & .2 & .3 & .4 & 0 \\ 0 & .6 & 0 & .4 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Identify the recurrent and transient states, and decompose the recurent states into irreducible closed sets.

(b) Compute, for each recurrent state, its period.

$\{1,5\}$, $\{2,4\}$ recurrent irreducible closed subsets (since they are finite)

$\{3\}$ is transient since $3 \to 1$ by $1 \not\to 3$.

period of $\{1,5\}$ is 2 as can only return in even steps.

period of $\{2,4\}$ is 1 as return possible in 1 step.

2. Consider the Markov chain on $\{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{bmatrix} .7 & 0 & .3 & 0 \\ 0 & .5 & 0 & .5 \\ .6 & 0 & .4 & 0 \\ 0 & .4 & 0 & .6 \end{bmatrix}.$$

(a) Find the stationary distribution(s) of the chain.

(b) Find $\lim_{n\to\infty} P^n$.

(b) $P^n \to \begin{bmatrix} \pi_1(1) & 0 & \pi_1(3) & 0 \\ 0 & \pi_2(2) & 0 & \pi_2(4) \\ \pi_1(1) & 0 & \pi_1(3) & 0 \\ 0 & \pi_2(2) & 0 & \pi_2(4) \end{bmatrix}$

By ergodic Theorem.

$\{1,3\}$, $\{2,4\}$ are closed irreducible recurrent sets, which are aperiodic and so each subset has a unique stat dist π_1, π_2 concentrated on them.

$\pi_1(1) = (.7)\pi_1(1) + (.6)\pi_1(3) \Rightarrow \pi_1(1) = 2/3$, $\pi_1(3) = 1/3$

$\pi_2(2) = (.5)\pi_2(2) + (.4)\pi_2(4) \Rightarrow \pi_2(2) = 4/9$, $\pi_2(4) = 5/9$

(a) $\Rightarrow \boxed{\pi = \alpha\pi_1 + (1-\alpha)\pi_2 \text{ where } 0 \le \alpha \le 1}$

3. Suppose brands A, B and C have brand loyalty of .6, .8 and .4 respectively. That is, someone who buys A in one week, will buy A again the following week with chance .7, or pick one of the other two brands with chance .3, choosing between B and C with equal probability.

(a) Find the transition matrix P for this Markov chain.

(b) Verify that $v = (3, 6, 2)$ is a left eigenvector of P with eigenvalue 1. What is the proportion of time that someone buys brand A or B?

(c) What is the expected time between purchases of brand A?

(a) $P = \begin{bmatrix} .6 & .2 & .2 \\ .1 & .8 & .1 \\ .3 & .3 & .4 \end{bmatrix}$

(b) $\pi = \left(3/11, 6/11, 2/11 \right)$

proportion buys A or B = $\pi(A) + \pi(B)$

$= 9/11$

(c) $E_A[T_A] = \pi(A)^{-1}$

$= 11/3$.

4. A bank classifies loans as paid in full (state 1), in good standing (state 2), in arrears (state 3), or as a bad debt (state 4). Loans move between these categories according to the transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .1 & .8 & .1 & 0 \\ .1 & .4 & .4 & .1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) What is the chance that a loan in good standing is eventually paid in full?

(b) How much (expected) time will a loan in good standing take to be either paid in full or be classified as bad debt?

(a) Want $u_{21} = P_2(T_1 < T_4)$. absorbtion probability.

(b) Want $S_{22} + S_{23}$ = expected time until absorbtion starting in 2.

$(I-Q)^{-1} = \begin{pmatrix} .2 & -.1 \\ -.4 & .6 \end{pmatrix}^{-1} = \frac{1}{.12 - .04} \begin{pmatrix} .6 & .1 \\ .4 & .2 \end{pmatrix}$

$R = \begin{pmatrix} .1 & 0 \\ .1 & .1 \end{pmatrix}$ $\quad (I-Q)^{-1}R = \frac{100}{8} \begin{pmatrix} .06 + .01 & .01 \\ .04 + .02 & .02 \end{pmatrix} \Rightarrow \boxed{u_{21} = 7/8}$

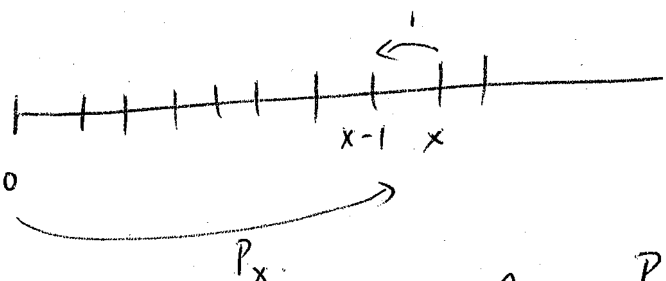
$\boxed{S_{22} + S_{23} = \frac{100}{8}(.7) = \frac{70}{8}}$

5. Consider the following Markov chain on $\{0, 1, 2, \ldots\}$ where $p(x, x-1) = 1$ for $x \geq 1$ and $p(0, x) = p_x$ for $x \geq 0$.

(a) When is the chain recurrent?

(b) Show that the chain is positive recurrent if and only if $\sum_{x \geq 0} x p_x < \infty$.

(c) In the case the chain is positive recurrent, identify the stationary distribution(s).



Note $\sum_{x=0}^{\infty} p(0,x) = \sum_{x=0}^{\infty} P_x = 1$.

(a) chain is irreducible.

$$\rho_{00} = P_0(T_0 < \infty) = \sum_{k \geq 1} P_0(T_0 = k) = \sum_{k \geq 1} P_{k-1} = 1$$

$\Rightarrow$ 0 recurrent always!

(b) $E_0(T_0) = \sum k P_0(T_0 = k) = \sum k P_{k-1} < \infty$ iff 0 is positive recurrent and so whole chain pos. recurrent.

(c) $\pi(0) = \pi(0) P_0 + \pi(1) \Rightarrow \pi(1) = \pi(0)(1 - P_0)$

$\pi(1) = \pi(0) P_1 + \pi(2) \Rightarrow \pi(2) = \pi(0)(1 - P_0 - P_1)$

$\vdots$

$\Rightarrow \pi(k) = \pi(0)(1 - P_0 - P_1 - P_2 - \cdots - P_{k-1})$

$$\Rightarrow \sum_{k \geq 0} \pi(k) = 1 = \pi(0) \sum_{k \geq 0} \left(1 - \sum_{\ell=0}^{k-1} P_\ell\right) = \pi(0) \sum_{k \geq 0} \sum_{\ell \geq k} P_\ell = \pi(0) \sum_{k \geq 1} k P_{k-1}$$

$$\Rightarrow \pi(0) = \frac{1}{\sum_{k \geq 1} k P_{k-1}} \Rightarrow \boxed{\pi(k) = \left(1 - \sum_{\ell=0}^{k-1} P_\ell\right) \Big/ \sum_{k \geq 1} k P_{k-1} \quad k \geq 0}$$

Note: $\sum_{\ell=0}^{-1} P_\ell = 0$ by convention