Order of Results in Class

Definitions

Definition: Markov property, and its implications

Computations: Chapman-Kolmogorov property, Matrix formulation, n-step distribution

Example: Two state Markov chain; asymptotics

Lemma: $p^{(n)}(x,y) = \sum_{m=1}^{n} P_x(T_y = m)p^{(n-m)}(y,y)$ for $n \geq 1$.

Recurrence/Transience

Definition: $x \to y$ if $\rho_{x,y} = P_x(T_y < \infty) > 0$.

Lemma: $\rho_{x,y} > 0 \Leftrightarrow$ there is ks.t. $p^{(k)}(x,y) > 0$.

Definition: x is recurrent if $\rho_{x,x} = 1$ and transient otherwise, if $\rho_{x,x} < 1$.

Proposition: If $x \to y$ and $y \not\to y$, then x is transient.

Computation: Distribution of $N(y)$, number of returns to y. $G(x,y) = E_x[N(y)] = \sum_{n=1}^{\infty} p^{(n)}(x,y)$.

Theorem[Characterization in terms of expected number of returns]: Let y be a transient state. Then $P_x(N(y) < \infty) = 1$ and $G(x,y) = \rho_{x,y}(1 - \rho_{y,y})^{-1} < \infty$ for all $x, y \in \Sigma$. On the other hand, let y be a recurrent state. Then, $P_y(N(y) = \infty) = 1$ and $G(y,y) = \infty$. Also, $P_x(N(y) = \infty) = \rho_{x,y}$. If $\rho_{x,y} = 0$, then $G(x,y) = 0$, otherwise if $\rho_{x,y} > 0$, then $G(x,y) = \infty$.

Lemma: For a finite state Markov chain, there is at least one recurrent state.

Proposition: Let x be recurrent, and $x \to y$. Then, y is recurrent and $\rho_{x,y} = \rho_{y,x} = 1$.

Decomposition of State space

Definition: a closed set C is a set of states such that $\rho_{x,y} = 0$ if $x \in C$ and $y \notin C$.

Lemma: C is closed if and only if $p(x,y) = 0$ when $x \in C$ and $y \notin C$.

Definition: an irreducible closed set C is a closed set such that $x \to y$ for all choices $x, y \in C$. An irreducible Markov chain is one where $x \to y$ for all $x, y \in \Sigma$.

Theorem: In an irreducible closed set, either all states are transient or all states are recurrent.

Corollary: In an irreducible closed finite set, all states are recurrent.

Theorem[Decomposition]: $\Sigma = \Sigma_T \cup \Sigma_R$ where Σ_T are the transient states and Σ_R are the recurrent states. If $\Sigma_R \neq$, then $\Sigma_R = \cup_i C_i$ where $\{C_i\}$ are disjoint closed irreducible sets of recurrent states.

Example: Birth-Death Markov chains.

Absorption Probabilities

Definition: Let $u_{i,k}$ be the probability starting from a transient state i that k is the first recurrent state hit. That is, when absorbed into Σ_R, k is the first recurrent state hit.

Definition: Let $s_{i,j}$ be the mean time in transient state j, when starting from transient state i, up to the absorption into Σ_R.

Computation: $U = (I - Q)^{-1}R$ and $S = (I - Q)^{-1}$

Stationary distributions

Definition: π=stationary distribution

Lemma: Distribution of X_n is independent of $n \Leftrightarrow$ Initial distribution is a stationary distribution.

Definition: Reversible distribution.

Lemma: A reversible distribution is a stationary distribution.

Example: Birth-Death Markov chains

Frequency Computation

Theorem: $N_n(y)/n \to 1_{T_y<\infty}/E_y[T_y]$ with probability 1. Also, $G_n(x,y)/n \to \rho_{xy}/E_y[T_y]$.

Corollary: Let C be an irreducible closed set of recurrent states. Then, $G_n(x,y)/n \to 1/E_y[T_y]$ for $x, y \in C$. Also, if $P(X_0 \in C) = 1$, then $N_n(y)/n \to 1/E_y[T_y]$ for $y \in C$ with probability 1.

Positive and Null Recurrence

Definition: Positive and Null recurrence.

Theorem: If y is transient or null recurrent, then $G_n(y,y)/n \to 0$. But, if y is positive recurrent, then $G_n(y,y)/n \to 1/E_y[T_y] > 0$.

Theorem: If x is positive recurrent and $x \to y$, then y is positive recurrent.

Corollary: An irreducible chain is all transient, all null recurrent or all positive recurrent.

Theorem: If C is a finite closed set of states, then there is at least one positive recurrent state.

Transience, Null and Positive Recurrence and Stationary Distributions

Theorem: Let π be a stationary distribution. Then $\pi(x) = 0$ for x which is transient or null recurrent.

Corollary: A stationary distribution π is supported on positive recurrent states. Or, stationary distributions do not exist for chains without any positive recurrent states.

Frequency Interpretation of Stationary Distributions

Theorem: An irreducible positive recurrent Markov chain has a unique stationary distribution given by $\pi(x) = 1/E_x[T_x]$.

Corollary: An irreducible chain is positive recurrent $\Leftrightarrow$ it has a stationary distribution.

Corollary: An irreducible finite state space chain has a unique stationary distribution.

Corollary: For an irreducible positive recurrent chain, $N_n(x)/n \to \pi(x)(= 1/E_x[T_x])$ with probability 1.

Example: Birth-Death Markov chains

Reducibility

Theorem: Let Σ_p be the positive recurrent states. Then, (1) if $\Sigma_p = \emptyset$, there is no stationary distribution. (2) If Σ_p is non-empty and irreducible, then there exists a unique stationary distribution. (3) If Σ_p is non-empty but reducible, then there exists an infinite number of stationary distributions.

Periodicity and Convergence

Theorem: For an irreducible, positive recurrent and aperiodic chain, $p^{(n)}(x,y) \to \pi(y)$. Also, for an irreducible positive recurrent chain with period $d > 1$, we have, for x and y, that there is $0 \leq r < d$ such that $p^{(n)}(x,y) = 0$ unless n is of the form $n = md + r$ for $m \geq 0$, and then $\lim_{m\to\infty} p^{(md+r)}(x,y) = d\pi(y)$.

Examples: Two state-chain, etc.