8.6 (a) Covariance for BM: $E[X_s X_t] = E[X_s(X_t - X_s)] + E[X_s^2]$

$$= (E X_s)(\overset{0}{\cancel{E X_t - X_s}}) + s \quad \text{if } s<t.$$

$E[X_s^2] = s$. This identifies Γ.

(b) $t_1 Y_1 + \cdots + t_n Y_n$

$$= X_1\left(\sum_{k=1}^{n} t_k a_{k1}\right) + \cdots + X_n\left(\sum_{k=1}^{n} t_k a_{kn}\right)$$

$$\Rightarrow E\left[e^{t_1 Y_1 + \cdots + t_n Y_n}\right] = e^{\left(\sum_{k=1}^{n} t_k a_{k1}\right)^2/2} \cdots e^{\left(\sum_{k=1}^{n} t_k a_{kn}\right)^2/2}$$

$$= e^{\sum_{\ell=1}^{n} \sum_{i,j} \frac{t_i t_j a_{i\ell} a_{j\ell}}{2}}$$

Now, $E[Y_i Y_j] = E[(a_{i1} X_1 + \cdots + a_{in} X_n)(a_{j1} X_1 + \cdots + a_{jn} X_n)]$

$$= \sum_{\ell} a_{i\ell} a_{j\ell}$$

$$\Rightarrow E\left[e^{t_1 Y_1 + \cdots + t_n Y_n}\right] = e^{\sum_{i,j=1}^{n} \frac{t_i t_j E[Y_i Y_j]}{2}}$$

$$= e^{\frac{\langle \vec{t}, \Gamma \vec{t} \rangle}{2}} \quad \text{where } \vec{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}.$$

(c) If $E Y_i Y_j = 0$ all $i \neq j$

Then $E\left[e^{t_1 Y_1 + \cdots + t_n Y_n}\right] = e^{\sum_{i=1}^{n} \frac{t_i^2 E[Y_i^2]}{2}}$ factors

$\Rightarrow Y_1, \ldots, Y_n$ independent $N(0, E(Y_i)^2)$ r.v.'s $i=1,\ldots,n$

8.8 $\langle Y_{s_1}, \ldots, Y_{s_n} \rangle$ jointly normal with mean 0 (using 8.5).

And, $Y_{s_i} = s_i X_{1/s_i} \sim N(0, s_i)$

To use 8.6 c, need to identify covariances: $E[Y_{s_i} Y_{s_j}] = s_i s_j E[X_{1/s_i} X_{1/s_j}]$

$= s_i s_j \frac{1}{s_j}$ if $s_i < s_j$

$= s_i$ same as for BM!

Here, the covariance matrices are the same, and so MGF same, and so distribution is the same as BM

$\Rightarrow \{Y_s : s \geq 0\}$ has the indep increments + distributional property of BM. Also, Y_s is continuous, even at $s=0$

$P(|Y_s| > \varepsilon) \leq \frac{E Y_s^2}{\varepsilon^2} = \frac{s}{\varepsilon^2} \to 0$

8.10 (a) $Z_t - Z_s = X_t - X_s + Y_t - Y_s$.

independent of $Z_s = X_s + Y_s$ as $X.$, $Y.$ indep BM's.

$\Rightarrow$ independent increments.

$Z_t \sim N(0, 2t)$ as sum of indep normals.

$\Rightarrow \sigma^2 = 2$

(b) Zero set for Z_t has infinitely many elements

as $\frac{1}{\sqrt{2}} Z_t$ is a st. BM.