<u>Theorem</u> An irreducible pos. recurrent MC has a unique stationary distribution π given by $\pi(x) = 1/m_x$ for $x \in \Sigma$.

<u>proof</u>: We know from irreducibility and recurrence that

$$\lim_{n\to\infty} \frac{G_n(z,x)}{n} = \frac{1}{m_x} > 0 \quad \forall x, z \in \Sigma .$$

Now suppose π is a stat. dist. Then as $\pi(x) = \sum_z \pi(z) \frac{G_n(z,x)}{n}$ for $n \geq 1$, we can take limit to obtain

$$\pi(x) = \lim_{n\to\infty} \sum_z \pi(z) \frac{G_n(z,x)}{n} \overset{BCT}{=} \frac{1}{m_x} \sum_z \pi(z) = \frac{1}{m_x}$$

$\Rightarrow$ if π exists it is unique and equals $\pi(x) = \frac{1}{m_x}$ for $x \in \Sigma$.

To finish the proof, need to show that $\{ \frac{1}{m_x} : x \in \Sigma \}$ is (1) a dist (2) satisfies Balance equation.

<u>Case $|\Sigma| < \infty$</u> $\forall z,y \in \Sigma$, $\sum_x \frac{G_n(z,x)}{n} = 1$

$$\text{also} \quad \sum_x p^{(m)}(z,x) p(x,y) = p^{(m+1)}(z,y) .$$

$$\Rightarrow \sum_x \frac{G_n(z,x)}{n} p(x,y) = \frac{G_{n+1}(z,y)}{n} - \frac{p(z,y)}{n}$$

$\Rightarrow$ Taking limit $n \to \infty$, as the sums are finite,

$$\sum_x \frac{1}{m_x} = 1 \quad \text{and} \quad \sum_x \frac{1}{m_x} p(x,y) = \frac{1}{m_y} .$$

<u>Case $|\Sigma| = \infty$</u> Cannot take limits as not clear $\sum_x p(x,y)$ summable. Let then $\Sigma_1 \subset \Sigma$ be a finite subset. Then,

$$\sum_{x \in \Sigma_1} \frac{G_n(z,x)}{n} \leq 1 \quad \text{and} \quad \sum_{x \in \Sigma_1} \frac{G_n(z,x)}{n} p(x,y) \leq \frac{G_{n+1}(z,y)}{n} - \frac{p(z,y)}{n}$$

Take limits: $\sum_{x \in \Sigma_1} \frac{1}{m_x} \leq 1$ and $\frac{1}{m_y} \geq \sum_{x \in \Sigma_1} \frac{1}{m_x} p(x,y)$

Let now $\Sigma_1 \uparrow \Sigma$ to get $\sum_{x \in \Sigma} \frac{1}{m_x} \leq 1$ and $\sum_x \frac{1}{m_x} p(x,y) \leq \frac{1}{m_y}$

Now suppose second inequality is strict for some y. Then

$$\sum_y \frac{1}{m_y} > \sum_y \sum_x \frac{1}{m_x} p(x,y) = \sum_x \frac{1}{m_x} \sum_y p(x,y)$$

$$= \sum_x \frac{1}{m_x}$$

As LHS = RHS, obtain contradiction

$\Rightarrow \{ \frac{1}{m_x} : x \in \Sigma \}$ satisfies Balance equations.

Now call $\sum_x \frac{1}{m_x} = c \leq 1$ [$c > 0$ as chain is pos. recurrent]

and define $\pi'(x) = \frac{1}{c m_x}$. Then π' is a distribution satisfying Balance equations — hence a stat. dist but by uniqueness must have

$\pi'(x) = \frac{1}{c m_x} = \frac{1}{m_x} \quad \forall x \in \Sigma. \Rightarrow c = 1$ //.