HW 2 Math 523a Fall 2016

Due Sep. 16, 2016

1. (Folland 5). If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, show that $\mathcal{M}$ is the union of the σ-algbras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. Hint: Show that the latter object is a σ-algebra.

2. (part of Folland 8). Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $\{E_j\}_{j\geq 1}$ be sets in $\mathcal{F}$. Show that $\mu(\liminf E_n) \leq \liminf_{n\to\infty} \mu(E_n)$.

3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $\{E_j\}_{j\geq 1}$ be sets in $\mathcal{F}$. Show that

$$\sum_{j\geq 1} \mu(E_j) < \infty \;\Rightarrow\; \mu(\limsup_n E_n) = 0.$$

4. Let $(\Omega, \mathcal{F}, \mu)$ be a **probability** space.

(a) Suppose $\{B_n\}_{n\geq 1}$ are sets in $\mathcal{F}$ such that $\mu(B_n) = 1$ for all $n \geq 1$. Show that $\mu(\cap_{n\geq 1} B_n) = 1$.

(b) Suppose $\{B_n\}_{n=1}^N$ are sets in $\mathcal{F}$ such that $\sum_{n=1}^N \mu(B_n) > N - 1$. Show that $\mu(\cap_{n=1}^N B_n) > 0$.

5. (Folland 12). Let $(\Omega, \mathcal{F}, \mu)$ be a **finite** measure space. Note that the 'symmetric difference' $E \triangle F = (E \setminus F) \cup (F \setminus E)$.

(a) Show, if $E, F \in \mathcal{F}$ and $\mu(E \triangle F) = 0$, then $\mu(E) = \mu(F)$.

(b) Say that $E \sim F$ if $\mu(E \triangle F) = 0$. Show that $\sim$ is an equivalence relation on $\mathcal{F}$.

(c) For $E, F \in \mathcal{F}$, define $\rho(E, F) = \mu(E \triangle F)$. Show that ρ defines a metric on the space $\mathcal{F} \setminus \sim$ of equivalence classes.