An introduction to scaling limits in interacting particle systems

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Outline

- Hydrodynamics of the ‘bulk’ mass
- ‘Replacement’ averaging principle methods
  ‘Entropy’ GPV method
  ‘Yau’s’ method
- Fluctuations of the ‘bulk’ mass and ‘occupation times’

Replacement

The main work to establish ‘hydrodynamics’ is the ‘replacement’ estimate to close the discrete evolution equations.

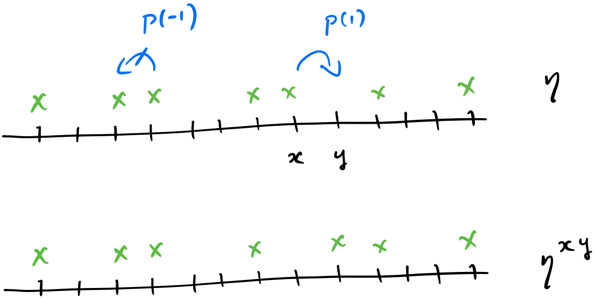
–This estimate is of its own interest, and may have application in other settings.

We will discuss two main techniques which have broad validity:

- ‘entropy’ method of GPV (1988), and
- ‘relative entropy’ method of Yau (1991).

Exclusion model

Recall the simple exclusion process on $\mathbb{T}_N^d = \mathbb{Z}^d / N\mathbb{Z}^d$ consists of a collection of continuous time RW's, with jump probabilities $p(x, y)$ going from x to y, where jumps to occupied locations are suppressed.



Its generator is

$$L_{SE}f(\eta) = \sum_{x,y} \big(f(\eta^{xy}) - f(\eta)\big)\eta(x)(1-\eta(y))p(y-x)$$

–Invariant measures include the Bernoulli product measures

$$\mu_\rho = \prod_x \mathrm{Bern}(\rho)$$

As before, we will start from initial configurations distributed according to a ‘local equilibrium’ measure

$$\mu^N = \prod_x \mathrm{Bern}(\rho_0(x/N))$$

where $\rho_0 : \mathbb{T}^d \to [0, 1]$.

Recall that we are speeding up time by N^{θ} and the grid spacing is $1/N$, and

$$\eta_t^N(x) = \eta_{N^{\theta}t}(x)$$

–Recall also the empirical measure

$$\pi_t^N = \frac{1}{N^d} \sum_x \eta_t^N(x) \delta_{x/N}.$$

Main aim

Let h be a local function,

e.g. $h(\eta) = \eta(0)\big(1 - \eta(1)\big)$ in $d = 1$, etc.

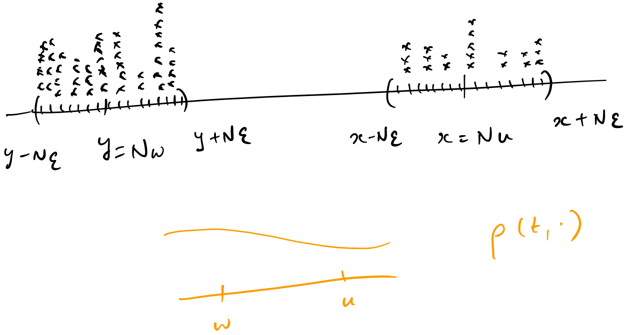
–Our goal is to approximate

$$h\big(\eta_t^N(x)\big)$$

by

$$E_{\mu_{\rho(t,u)}}[h] =: H\big(\rho(t,u)\big)$$

where $x = Nu$.



$y - N\varepsilon$
$y = Nw$
$y + N\varepsilon$
$x - N\varepsilon$
$x = Nu$
$x + N\varepsilon$
$\rho(t,\cdot)$
w
u

Recall that

$$\eta^{(N\epsilon)}(x) = \frac{1}{(2N\epsilon+1)^d} \sum_{|y|\leq \ell} \eta(x+y)$$
$$= \langle i_\epsilon, \pi_s^N \rangle$$

can be written in terms of π_t^N.

–Here, as before,

$$i_\epsilon(u) = (2\epsilon)^{-1} 1(|u| \leq \epsilon).$$

‘Entropy’ method

We will show in a sense that

$$h\big(\eta_t^N(x)\big) \sim H\big(\eta_t^{(N\epsilon)}\big)$$

from which things follow:

Indeed, as $\eta_{N^\theta t}^{(N\epsilon)}(x)$ is a macroscopic average,
in an ϵ window,
it will be close to $\rho(t, u)$, as discussed last time.

Remark

The ‘entropy’ method works well when

- the setting is translation-invariant
- the dynamics is reversible ($\theta = 2$)

–Elements of the ‘entropy’ method replacement though will be useful in the asymmetric setting as well,
e.g. the upcoming 1-block lemma will hold.

Recall τ_x denotes a shift by x.
Let J be a test function.

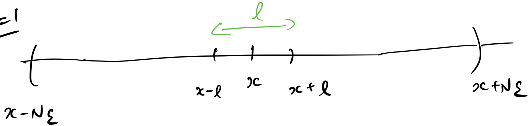
Theorem. We have

$$\limsup_{\epsilon\downarrow 0}\limsup_{N\uparrow\infty} E_{\mu^N}\Big[\Big|\int_0^T \frac{1}{N^d}\sum_x J\big(\tfrac{x}{N}\big)\ \tau_x V_{N\epsilon}(\eta_s^N)ds\Big] = 0$$

where

$$V_\ell(\eta) = \Big\{h(\eta) - H\big(\eta^{(\ell)}(0)\big)\Big\}.$$

$d=1$



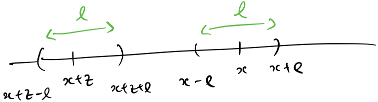
- $\tau_x h(\eta) \approx \dfrac{1}{2\ell+1} \sum_{|y|<\ell} \tau_{x+y} h(\eta)$

  $\approx H(\eta^{(\ell)}(x))$ "1-block"

- $H(\eta^{(\ell)}(x)) \approx H(\eta^{(N\varepsilon)}(x))$ "2-block"

  via $\eta^{(\ell)}(x) \approx \eta^{(\ell)}(x+z)$

  where $|z| < N\varepsilon$.



Introducing scale $1 << \ell << N$

Write

$$
\begin{aligned}
&\frac{1}{N^d}\sum_x J\big(\tfrac{x}{N}\big)\ \tau_x V^{N\epsilon}(\eta) \\
&= \ \frac{1}{N^d}\sum_x J\big(\tfrac{x}{N}\big)\ \Big\{\tau_x h(\eta) - \frac{1}{(2\ell+1)^d}\sum_{|z|\le\ell}\tau_{z+x}h(\eta)\Big\} \\
&+ \ \frac{1}{N^d}\sum_x J\big(\tfrac{x}{N}\big)\ \Big\{\frac{1}{(2\ell+1)^d}\sum_{|z|\le\ell}\tau_{z+x}h(\eta) - H\big(\eta^{(\ell)}(x)\big)\Big\} \\
&+ \ \frac{1}{N^d}\sum_x J\big(\tfrac{x}{N}\big)\ \Big\{H\big(\eta^{(\ell)}(x)\big) - H\big(\eta^{(N\epsilon)}(x)\big)\Big\}.
\end{aligned}
$$

The first line on RHS introduces more averaging.

–By smoothness of J,

it is of order $O(\ell^d/(N\epsilon))$.

The second term, bringing an absolute value inside the sum, is bounded by

$$\frac{\|J\|_{L^\infty}}{N^d} \sum_x \tau_x W_\ell(\eta)$$

where

$$W_\ell(\eta) = \Big| \frac{1}{(2\ell+1)^d} \sum_{|y|\le\ell} \tau_y h(\eta) - H(\eta^{(\ell)}(0)) \Big|.$$

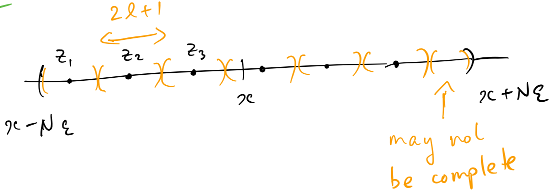
While in the third term, as H is Lipschitz, we bound by

$$\frac{\|J\|_{L^\infty}}{N^d}\sum_x \left|H(\eta^{(\ell)}(x)) - H(\eta^{(N\epsilon)}(x)\right| \leq \frac{\|J\|_{L^\infty}}{N^d}\sum_x \left|\eta^{(\ell)}(x) - \eta^{(N\epsilon)}(x)\right|.$$

We may further write the $N\epsilon$-window term $\eta^{(N\epsilon)}(x)$ in terms of an average of ℓ-window terms $\eta^{(\ell)}(x + z_i)$ for $i = 1, \ldots, M := (2N\epsilon + 1)^d/(2\ell + 1)^d$:

$$\eta^{(N\epsilon)}(x) = \frac{(2\ell + 1)^d}{(2N\epsilon + 1)^d} \sum_{i=1}^{M} \eta^{(\ell)}(x + z_i) + O\left(\frac{\ell^d}{N^d}\right)$$

$\ell = 1$



Finally, ‘omitting’ ℓ-blocks near x,

we bound the third term:

$$\sup_{2\ell \le |y| \le N\epsilon} \frac{C}{N^d} \sum_{x} |\eta^{(\ell)}(x) - \eta^{(\ell)}(x+y)| + O(\ell^d / N^d)$$

1-block lemma

Let P_t^N be the semigroup for the N^θ speeded up process, and let μ_ρ be a reference measure.

Denote the probability density

$$f_T^N := \frac{1}{T}\int_0^T \frac{d\mu^N P_s^N}{d\mu_\rho} ds.$$

With respect to the second term before, integrating in time, taking expectation,

$$E_{\mu^N}\Big[\int_0^T \frac{1}{N^d}\sum_x \tau_x W_\ell(\eta_s^N) ds\Big]$$

$$= T\ E_{\mu_\rho}\big[f_T^N(\eta)\ \frac{1}{N^d}\sum_x \tau_x W_\ell(\eta)\big].$$

<u>Lemma</u> (1-block) We have

$$\lim_{\ell\uparrow\infty}\lim_{N\uparrow\infty} E_{\mu_\rho}\big[f_T^N(\eta)\ \frac{1}{N^d}\sum_x \tau_x W_\ell(\eta)\big] = 0.$$

2-block lemma

Similarly, w.r.t. the third term, integrating in time and taking expectation, we show the following.

Lemma (2-block) We have

$$\lim_{\ell\uparrow\infty}\lim_{\epsilon\downarrow 0}\lim_{N\uparrow\infty}\sup_{2\ell\le|y|\le 2N\epsilon} E_{\mu_\rho}\Big[f_T^N(\eta)\,\frac{1}{N^d}\sum_x\big|\eta^{(\ell)}(x)-\eta^{(\ell)}(x+y)\big|\Big]=0.$$

Measuring f_T^N

1. Consider the relative entropy of μ^N with respect to μ_ρ:

$$H(\mu^N;\mu_\rho) = E_{\mu^N}\big[\log\frac{d\mu^N}{d\mu_\rho}\big]$$

$$= \sum_x \Big[\rho_0(x/N)\log\rho_0(x/N)/\rho + (1-\rho_0(x/N))\log\frac{1-\rho_0(x/N)}{1-\rho}\Big]$$

This is $O(N^d)$ as we are on the torus $\mathbb{T}_N^d$.

2. At time t, for the N^θ-speeded up process,
   with semigroup P_t^N,

the rate of change is

$$\frac{d}{dt} H(\mu^N P_t^N; \mu_\rho) = N^\theta E_{\mu_\rho} \Big[\frac{d\mu^N P_t^N}{d\mu_\rho} L \log \frac{d\mu^N P_t^N}{d\mu_\rho} \Big].$$

3. A calculation gives

$$E_{\mu_\rho}\Big[\frac{d\mu^N P_t}{d\mu_\rho} L \log \frac{d\mu^N P_t}{d\mu_\rho}\Big] \le -2D\Big(\sqrt{\frac{d\mu^N P_t^N}{d\mu_\rho}}\Big)$$

where, for Exclusion,

$$\begin{aligned} D(h) &= E_{\mu_\rho}\big[h(-Lh)\big] \\ &= \frac{1}{4}\sum_{x,y} s(y-x) E_{\mu_\rho}\Big[\big(h(\eta^{xy}) - h(\eta)\big)^2\Big]. \end{aligned}$$

–Here, $s(z)$ is the symmetrization $\big(p(z)+p(-z)\big)/2$

4. Then $(**)$,

$$\frac{d}{dt}H(\mu^N P_t^N;\mu_\rho) \le -2N^\theta D\Big(\sqrt{\frac{d\mu^N P_t^N}{d\mu_\rho}}\Big)$$

and

$$\begin{aligned} H(\mu^N P_T^N;\mu_\rho) + 2N^\theta \int_0^T D\Big(\sqrt{\frac{d\mu^N P_s^N}{d\mu_\rho}}\Big)ds \\ \le \ H(\mu^N;\mu_\rho) \ \le \ CN^d. \end{aligned}$$

**Uses $a\log(b/a) \le \sqrt{a}\big[\sqrt{b}-\sqrt{a}\big]$ for $a, b > 0$.

5. Abbreviate and recall

$$I(h) = D(\sqrt{h}), \quad \text{and} \quad f_T^N = \frac{1}{T}\int_0^T \frac{d\mu^N P_s^N}{d\mu_\rho} ds.$$

–By convexity of Dirichlet form,

$$I\big(f_T^N\big) \le CTN^{d-\theta}.$$

Sketch: 1-block lemma

The idea is, when localized in a ℓ-block,

the Dirichlet form of $f = f_T^N$ vanishes in the $N \uparrow \infty$ limit.

This means f is roughly constant.

Ergodicity w.r.t. μ_ρ now applies.

Highlights

A. Write

$$E_{\mu_\rho}\Big[f(\eta)\cdot\frac{1}{N^d}\sum_x \tau_x W_\ell(\eta)\Big] = E_{\mu_\rho}\Big[Av(f)\cdot W_\ell(\eta)\Big] = E_{\mu_\rho}\Big[f_\ell(\eta)W_\ell(\eta)\Big].$$

where

$$Av(f) = \frac{1}{N^d}\sum_x \tau_x f(\eta) \text{ and } f_\ell(\eta) = E_{\mu_\rho}[Av(f)|\mathcal{F}_\ell].$$

–Here, $\mathcal{F}_\ell = \sigma\{\eta(x) : |x| \le \ell\}$.

B. Consider the 'ℓ-block' Dirichlet form

$$I_\ell(w) = \sum_{x,y:|x|,|x+y|\le\ell} I_{x,x+y}(w)$$

where

$$I_{x,x+y}(w) = \frac{1}{4} s(y-x) E_{\mu_\rho}\Big[\big(\sqrt{w}(\eta^{xy}) - \sqrt{w}(\eta)\big)^2\Big].$$

By translation-invariance,

$$
\begin{aligned}
I_{x,x+y}(w) &= \frac{1}{N^d} \sum_{z \in \mathbb{T}_N^d} I_{z,z+y}(w) \\
&\le \frac{1}{N^d} \sum_{z,z'} I_{z,z'}(w) \\
&= \frac{1}{N^d} I(w).
\end{aligned}
$$

–Since p is finite-range, we have

$$
I_\ell(w) \le C\ell^d N^{-d} I(w).
$$

Then, by convexity, and $I(f) \leq CN^{d-\theta}$,
we have

$$\begin{aligned} I_\ell(f_\ell) &\leq I_\ell(Av(f)) \\ &\leq C\ell^d N^{-d} I(Av(f)) \\ &\leq C\ell^d N^{-d} I(f) \\ &\leq C\ell^d N^{-\theta}. \end{aligned}$$

C. Considering limit points as $N \uparrow \infty$, need only show

$$\lim_{\ell \uparrow \infty} \sup_{I_\ell(f)=0} E_{\mu_\rho}\big[f(\eta) V_\ell(\eta)\big] = 0.$$

But, looking at the form of I_ℓ,
 if $I_\ell(f) = 0$,
conclude f is constant
 on configurations on $\{-\ell, \cdots, \ell\}$ such that $\eta^{(\ell)}(0) = a$.

–Here, the values $0 \leq a \leq 1$ (in the Exclusion process).

So, it would be enough to show

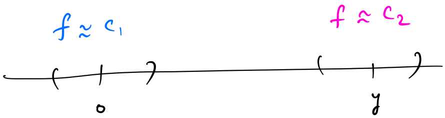
$$\lim_{\ell\uparrow\infty} \sup_{0\le a\le 1} E_{\mu_\rho}\Big[\Big|\frac{1}{(2\ell+1)^d}\sum_{|x|\le\ell}\tau_x h(\eta) - H(a)\Big|\Big|\eta^{(\ell)}(0) = a\Big] = 0.$$

–Recall $H(a) = E_{\mu_a}[h]$.

–The measure μ_ρ is product.

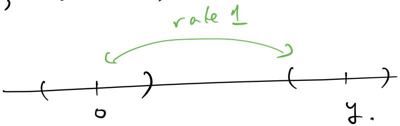
–At this point, the last limit can be seen via local central limit theorems, for instance.

A cartoon about 2-block lemma

- 

using 1-block proof.

- But, $|y| \le \varepsilon N$, so ℓ-blocks not far away



Introduce "communication" between $0 \leftrightarrow y$.

- Can show Dirichlet form of new process
$\hat{I}(f) \le C\varepsilon$ is small

Replacement by Yau's method

Recall our original framework. We start from μ^N and evolve the system in time scale $N^\theta t$.

–The variables

$$\left\{ \eta_t^N(x) : x \in \mathbb{T}_N^d \right\}$$

are governed by distribution $\mu_t^N := \mu^N P_t^N$.

These variables are not independent for $t > 0$,
even if the initial distribution is product.

However, suppose we believe that the system is close to the macroscopic picture with respect to solution $\rho(t, u)$.

–The idea is that μ_t^N should be 'close' to a product measure with means given by

$$\left\{\rho(t, x/N) : x \in \mathbb{T}_N^d\right\}.$$

Let ρ be a <u>smooth</u> solution of the hydrodynamic PDE, bounded away from 0 and 1,

for $0 \leq t \leq T$, for a $T > 0$

(T could be small, starting from smooth initial data).

Form, for $t \geq 0$,

$$\nu_t^N = \prod_x Bern(\rho(t, x/N)).$$

Note: ν_0^N is a local equilibrium measure.

Consider $d = 1$ Exclusion, but say p is asymmetric ($\theta = 1$).

Theorem. Suppose

$$H(\mu^N; \nu_0^N) = o(N).$$

Then,

$$\lim_{N \uparrow \infty} \frac{1}{N} H(\mu_t^N; \nu_t^N) = 0.$$

How does this imply hydrodynamics?

Consider the variational definition of entropy:

$$H(\mu;\nu) = \sup_F \big\{ E_\mu[F] - \log E_\nu[e^F] \big\}.$$

–From this, one can derive the inequality for event A:

$$\mu(A) \leq \frac{\log 2 + H(\mu;\nu)}{\log\left(1 + \frac{1}{\nu(A)}\right)}.$$

Applying with $\mu = \mu_t^N$ and $v = v_t^N$, and

$$A = \Big\{ \Big| \frac{1}{N} \sum_x J(x/N) \eta_t^N(x) - \frac{1}{N} \sum_x J(x/N) \rho(t, x/N) \Big| > \epsilon \Big\},$$

we need to show

$$v_t^N(A) \leq e^{-CN}.$$

–This is a consequence of large deviation estimates for independent variables.

Highlights

Consider a reference measure $\mu_{1/2}$.

Let

$$\begin{aligned}\psi_t^N(\eta) &= \frac{d\nu_t^N}{d\mu_{1/2}}(\eta) \\ &= \prod_x \left(\frac{\rho(t, x/N)}{1/2}\right)^{\eta(x)} \left(\frac{1-\rho(t, x/N)}{1-1/2}\right)^{1-\eta(x)}.\end{aligned}$$

Using the forward equation,
and some calculation($**$),
the derivative of relative entropy may be bounded:

$$\frac{d}{dt} H(\mu_t^N | \nu_t^N) \leq E_{\mu_t^N}\Big[\frac{N}{\psi_t^N(\eta)} L^* \psi_t^N(\eta) - \partial_t \log \psi_t^N \Big]$$

$**$Uses $a\big[\log b - \log a\big] \leq b - a$ for $a, b > 0$.

So, we have

$$
\begin{aligned}
&H(\mu_t^N; v_t^N) \\
&\quad \leq H(\mu^N; v_0^N) + \int_0^t E_{\mu_s^N}\Big[\frac{N}{\psi_s^N(\eta)} L^* \psi_s^N(\eta) - \partial_t \log \psi_s^N\Big] ds.
\end{aligned}
$$

If we can bound the integral by

$$
o(N) + \kappa \int_0^t H(\mu_s^N; v_s^N) ds,
$$

with small κ,

then we may conclude by Gronwall's lemma.

1. The integrand can be computed.

In the context of TASEP in $d = 1$,
the dominant term, divided by N, is in form

$$E_{\mu_t^N}\Big[\frac{1}{N}\sum_x \eta(x+1)(1-\eta(x))\frac{-\partial_x \rho(t,x/N)}{\rho(1-\rho)(t,x/N)} - \frac{1}{N}\sum_x \eta(x)\frac{\partial_t \rho(t,x/N)}{\rho(1-\rho)(t,x/N)} + \frac{1}{N}\sum_x \frac{\partial_t \rho(t,x/N)}{(1-\rho)(t,x/N)}\Big].$$

2. Replace, by 1-block Lemma,
   the terms

$$\eta(x+1)\big(1-\eta(x)\big) \text{ by } \eta^{(\ell)}(x)\big(1-\eta^{(\ell)}(x)\big)$$
$$\eta(x) \text{ by } \eta^{(\ell)}(x).$$

3. Recall

$$
\begin{aligned}
\partial_t \rho &= -\partial_x \big(\rho(1-\rho)\big) \\
&= (2\rho - 1)\partial_x \rho.
\end{aligned}
$$

Let

$$
F(m,\rho) = -\frac{m(1-m)}{\rho(1-\rho)} - m\frac{2\rho-1}{\rho(1-\rho)} + \frac{2\rho-1}{1-\rho}.
$$

Then,

$$
F(\rho,\rho) = F_m(\rho,\rho) = 0,
$$

and

$$
\left| F(\eta^{(\ell)}(x),\rho) - F(\rho,\rho) \right| \le C \left| \eta^{(\ell)}(x) - \rho \right|^2 .
$$

Hence, the expectation in previous slide is less than

$$E_{\mu_t^N}\Big[\frac{C}{N}\sum_x \Big|\eta^{(\ell)}(x) - \rho(t, x/N)\Big|^2\Big].$$

This can be bounded($**$), multiplying back by N, by

$$o(N) + \kappa \int_0^t H\big(\mu_s^N; v_s^N\big)\, ds$$

as desired.

**Use an entropy inequality,
and large deviations starting from v_t^N.

Remarks

1. In ‘Yau’s’ method, smoothness of the solution is needed.

   But, only the 1-block Lemma is used.

   So, in asymmetric Exclusion, the method is valid up to the time T that a discontinuity of solution presents.

Note: As a consequence, uniqueness of solution to the PDE is shown up to time T.

2. However, in the ‘entropy’ method, there is no limitation on the time T.

Both 1 and 2-blocks are used, limiting use to contexts with say diffusive scaling.

Note: But, as a consequence, one derives existence of a weak solution.

References

There are several good books on interacting particle systems:

- De Masi-Presutti: Mathematical methods for hydrodynamic limits 1991
- Kipnis-Landim: Scaling limits of interacting particle systems 1999
- Liggett: Interacting particle systems 1985
- Spohn: Large scale dynamics of interacting particles 1991

Thank you!