An introduction to scaling limits in interacting particle systems

Sunder Sethuraman (U. Arizona)

July 17 - 21, 2023

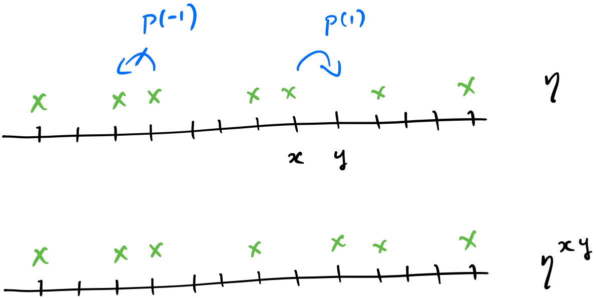
Stochastic methods in finance and physics IV
Crete workshop

Outline

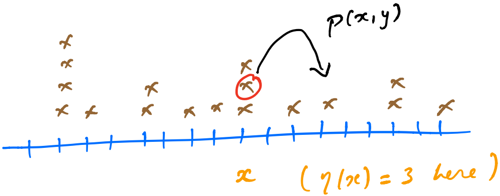
- Hydrodynamics of the ‘bulk’ mass
- ‘Replacement’ averaging principle methods
- Fluctuations of the ‘bulk’ mass and ‘occupation times’
  - Fluctuation field limits
  - Connections to occupation times, etc.

Exclusion interactions

Recall, the simple exclusion process on $\mathbb{Z}^d$ consists of a collection of continuous time RW's, with jump probabilities $p(x, y)$ going from x to y, where jumps to occupied locations are suppressed.



Zero-range interactions



"At x, a clock rings at rate $g(\eta(x))$. Then, a particle at random is selected, which moves to y with chance $p(x, y)$"

–Here, g is a nonnegative function, which specifies the interaction.

–When $g(k) \equiv k$, the process is that of independent random walks.

Hydrodynamic limit

Consider the $d \geq 1$ nearest-neighbor symmetric Exclusion process on $\mathbb{Z}^d$.

–Recall that $\eta_t^N = \eta_{N^2 t}$ with $\theta = 2$, and μ^N is an initial 'local' equilibrium measure associated to $\rho_0 : \mathbb{R}^d \to [0,1]$.

We have

$$\pi_t^N = \frac{1}{N^d} \sum_x \eta_t^N(x) \delta_{x/N} \Rightarrow \rho(t,u)du$$

where

$$\partial_t \rho = p(e) \Delta \rho.$$

–Here, $p(e) = 1/2d$.

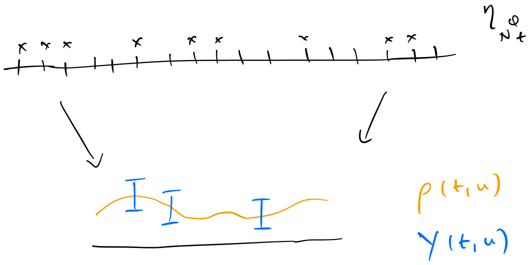
Fluctuations from the hydrodynamic limit

One might ask about the scale of the 'errors' in the hydrodynamic limit.

Define the fluctuation field

$$Y_t^N = \frac{1}{N^{d/2}} \sum_x \left(\eta_t^N(x) - E_{\mu^N}[\eta_t^N(x)] \right) \delta_{x/N}.$$

$d=1$



How does Y evolve?

We may look at the evolution of

$$Y_t^N(J) = \frac{1}{N^d} \sum_x J(x/N)\Big(\eta_t^N(x) - E_{\mu^N}[\eta_t^N]\Big)$$

for a test function J.

As before,

$$Y_t^N(J) = Y_0^N(J) + \int_0^t \left(\partial_t + N^\theta L\right) Y_s^N(J) ds + M_t^Y.$$

Compute

$$N^{\theta} L Y_t^N(J)\rangle \;=\; \frac{N^{\theta}}{N^{d/2}} \sum_x J(x/N) L\eta_t^N(x).$$

–Also, the time-derivative, using $\partial_t P_t^N = N^{\theta} P_t^N L$:

$$\partial_t Y_t^N(J)\rangle \;=\; \frac{-N^{\theta}}{N^{d/2}} \sum_x J(x/N) E_{\mu^N}[L\eta_t^N(x)].$$

–In $d = 1$,

$$L\eta_t^N(x) = \frac{1}{2}\Big\{\eta_t^N(x+1) - 2\eta_t^N(x) + \eta_t^N(x-1)\Big).$$

If we add these together,
after summing-by-parts,
we obtain

$$
\begin{aligned}
&\big(\partial_t + N^\theta L\big)\, Y_t^N(J) \\
&\quad = \frac{p(e)}{N^{d/2}} \sum_x \Delta J(x/N)\Big(\eta_t^N(x) - E_{\mu^N}[\eta_t^N(x)]\Big) + o(1) \\
&\quad = Y_t^N(\Delta J)\rangle + o(1).
\end{aligned}
$$

However, in the fluctuation field scaling,

the martingale

$$M_t^N = Y_t^N(J) - Y_0^N(J) - \int_0^t \left(\partial_t + N^\theta L\right) Y_s^N(J) ds$$

does not vanish.

Consider the ‘square’ martingale

$$(M_t^Y)^2 - N^\theta \int_0^t L\big(Y_s^N(J)\big)^2 - 2Y_s^N(J)LY_s^N(J)ds.$$

The time integral can be computed as

$$\int_0^t \frac{p(e)}{N^d} \sum_x \sum_{i=1}^d \sum_\pm \eta_s^N(x)(1-\eta_s^N(x \pm e_i)\Big(\partial_{x_i} J(x/N)\Big)^2 ds + o(1).$$

From the ‘GPV replacements’ already proved, this integral converges to

$$2p(e)\int_0^t\int \rho(s,u)\big(1-\rho(s,u)\big)\big|\nabla J\big|^2(u)du,$$

the quadratic variation of the martingale M_t^N limit.

With these calculations, we may formulate steps:

Step 1:

Show tightness of

$$\{Y_t^N : t \in [0, T]\} \ \text{in} \ D([0, T], \mathcal{S}'_d),$$

where $\mathcal{S}'_d$ is the space of tempered distributions.

Initially, at $t = 0$,

$$Y_0^N(J) = \frac{1}{\sqrt{N}} \sum_x J(x/N)\big(\eta_0(x) - \rho_0(x/N)\big)$$
$$\Rightarrow Y_0(J),$$

where the mean-zero Gaussian 'white noise' field Y_0 has covariance

$$E\big[Y_0(J)Y_0(K)\big] = \int J(u)K(u)\rho_0(u)(1 - \rho_0(u))du.$$

Step 2:

Identify limit points in terms of a unique ‘infinite dimensional BM’ process.

–If Y_t is a limit point, informally

$$dY_t = p(e)\Delta Y_t dt + \sqrt{2p(e)}dW_t$$

–Here, for $s \leq t$,
if we had started from an invariant measure μ_ρ
(so $\rho(t, u) \equiv \rho$), then

$$E\big[Y_t(J)Y_s(K)\big] = \rho(1-\rho)\int JT_{t-s}K$$

$$E\big[W_t(J)Y_s(K)\big] = \min\{s, t\}\int \nabla J \cdot \nabla K du$$

Martingale problem

More carefully, we conclude that

Y_t satisfies the martingale problem:

There is a unique distribution Q governing Y_t, concentrated on $C([0,T],\mathcal{S}'_d)$, such that

$$M_t(J) = Y_t(J) - Y_0(J) - p(e)\int_0^t Y_s(\Delta J)ds$$

$$N_t(J) = (M_t(J))^2 - 2p(e)\int_0^t \int \rho(s,u)(1-\rho(s,u))\big|\nabla J\big|^2 duds$$

are martingales.

–goes back to Holley-Stroock '79

Equilibrium fluctuations in zero-range models

In other models, without the ‘closing’ properties of symmetric exclusion, some type of ‘replacement’ will be needed. The theory is most robust, if we start in an invariant measure μ_ρ, where things are already interesting.

Consider nearest-neighbor symmetric Zero-range processes, with initial distribution μ_ρ.

As before

$$Y_t^N(J) = \frac{1}{N^{d/2}} \sum_x J(x/N)\big(\eta_t^N(x) - \rho\big)$$

satisfies a discrete evolution equation.

–Here,

$$N^\theta L Y_t^N(J) = \frac{p(e)}{N^{d/2}} \sum_x \Delta J(x/N) g(\eta_t^N(x)) + o(1).$$

Before, in the hydrodynamic scaling, $g(\eta_t^N(x))$ could be replaced by a homogenized function of the empirical measure.

–The difficulty now is that, in dividing by $N^{d/2}$, we have less room to manuever, and have to include more terms in the approximation.

Boltzmann-Gibbs replacement

Theorem. We have

$$\lim_{N\uparrow\infty} E_{\mu_\rho}\Big| \int_0^t \frac{1}{N^{d/2}} \sum_x J(x/N) \times \Big\{ g(\eta_s^N(x)) - \phi(\rho) - \phi'(\rho)\big(\eta_s^N(x) - \rho\big)\Big\} ds \Big|^2 = 0.$$

–Recall, that $\rho(t,u) \equiv \rho$, as we start from μ_ρ.

–Goes back to Brox-Rost '84

In a sense, to one more order,

$$\begin{aligned} g(\eta_t^N) &\sim E\Big[g(\eta_t^N(x))|\eta_{N^\theta t}^{(N\epsilon)}\Big] \\ &\sim \phi(\rho) + \phi'(\rho)\Big(\eta_{N^\theta t}^{(N\epsilon)} - \rho\Big). \end{aligned}$$

The associated martingale $M_t^N(H)$ has quadratic variation,

$$\int_0^t \frac{2p(e)}{N^d} \sum_x \left|\nabla H\right|^2 (x/N) g(\eta_s^N) ds$$
$$\sim 2p(e)\phi(\rho)\|\nabla H\|_{L^2}^2 t.$$

With the BG principle,
we obtain

$$dY_t(H) = p(e)\phi'(\rho)Y_t(\Delta H)dt + \sqrt{2p(e)\phi}(\rho)dW_t,$$

more precisely written in the martingale problem format.

–Here, W_t is as before.

Comments

1. There are only few results, starting out of the invariant measure, in more general symmetric systems.
   See Chang-Yau '92, and more recently Jara-Menezes 2018.

2. But, for fully asymmetric systems, in $d = 1$, there is now a wealth of results on various objects, including 'height function', via integrable probability,
   to KPZ class limits: Space $1/N$, time $N^{3/2}$, deviation $1/\sqrt{N}$.

3. For asymmetric systems, in $d \geq 2$, there are less results.
   Chang-Landim-Olla 2001, Landim-Olla-Varadhan 204, Caravenna-Sun-Zygouras '20, Chatterjee-Dunlap '20, Gu '21, Cannizzaro-Erhard-Toninelli '21

A look at ‘local’ statistics

Other statistics are of course of interest,
such as ‘current’ through a bond,
‘occupation time’ at a site, or
the motion of a distinguished particle, a ‘tagged’ particle, etc.

–To consider something different, let’s look at ‘occupation times’ with respect to Exclusion processes.

Basic problem

In the Exclusion process on $\mathbb{Z}^d$,
let $f : \Omega \to \mathbb{R}$ be a local function.
What is the behavior of

$$A_f(t) = \int_0^t f(\eta_s) ds$$

as $t \uparrow \infty$?

–We will start the process under an invariant measure μ_ρ.

Since μ_ρ is extremal, the process is ergodic with respect to time-shifts.

So, the a.s. law of large numbers holds:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\eta_s)ds = E_{\mu_\rho}[f].$$

Fluctuations of the occupation time

One can ask about the fluctuations. Are they diffusive and Gaussian?

–We will now concentrate on the centered function

$$f(\eta) = \eta(0) - \rho.$$

It turns out the answers depend on

- the dimension d,
- the structure of jump probability (e.g. symmetric or asymmetric), and
- the density ρ.

–There are still some open questions.

Variance

Let's try to compute the variance of $A_f(t)$:

$$
\begin{aligned}
Var(A_f(t)) &= E_{\mu_\rho}\Big[\Big(\int_0^t f(\eta_s)ds\Big)^2\Big] \\
&= 2\int_0^t \int_r^t E_{\mu_\rho}\big[f(\eta_s)f(\eta_r)\big]\,drds.
\end{aligned}
$$

By stationarity, since

$$E_{\mu_\rho}\big[f(\eta_s)f(\eta_r)\big] = E_{\mu_\rho}\big[f(\eta_{s-r})f(\eta_0)\big],$$

we have further

$$Var(A_f(t)) = 2\int_0^t (t-s)E_{\mu_\rho}\big[f(\eta_s)f(\eta_0)\big]ds.$$

Two point function

With

$$f(\eta) = \eta(0) - \rho,$$

the centered occupation variable,

we may compute

$$\begin{aligned} &E_{\mu_\rho}\big[(\eta_s(0) - \rho)(\eta_0(0) - \rho)\big] \\ &\quad = \rho(1-\rho)\Big\{E_{\mu_\rho}[\eta_s(0)|\eta_0(0) = 1] - E_{\mu_\rho}[\eta_s(0)|\eta_0(0) = 0]\Big\}. \end{aligned}$$

This can be viewed in terms of ‘coupling’:

-On the RHS, in the first expectation, the origin is occupied initially.

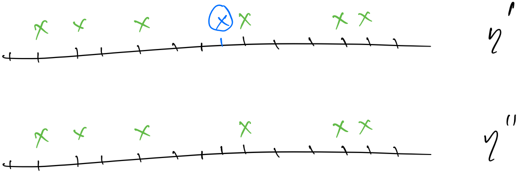
–In the second expectation, it is empty.

Basic coupling

Couple two copies of the Exclusion process,
starting from configurations $\eta' \geq \eta''$, such that

$\eta'(x) = \eta''(x)$ for all $x \neq 0$, and $\eta'(0) = 1$, while $\eta''(0) = 0$.

$d=1$



The coupled system (η'_t, η''_t) has generator

$$
\begin{aligned}
&\bar{L} f(\eta', \eta'') \\
&= \sum_{x,y} p(y-x) 1(\eta'(x) = \eta''(x) = 1)\big[f\big((\eta')^{xy}, (\eta'')^{xy}\big) - f\big(\eta', \eta''\big)\big] \\
&+ \sum_{x,y} p(y-x) 1(\eta'(x) = 1, \eta''(x) = 0)\big[f\big((\eta')^{xy}, \eta''\big) - f\big(\eta', \eta''\big)\big].
\end{aligned}
$$

Second-class particle

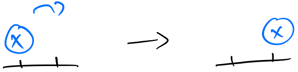
Here, the process η'_t, for times $t \geq 0$, majorizes η''_t.

–There is exactly one discrepancy, which we label R_t.

The dynamics of R_t is as follows:

It displaces $R_t \to R_t + z$
with rate

$$p(z)(1 - \eta''(R_t + z)) + p(-z)\eta''(R_t + z).$$



rate $p\,(1-\eta'(+))$

rate $q\,(1-\eta'(-))$

rate $q\,\eta'(+)$

rate $p\,\eta'(-)$

Then, the two-point function

$$
\begin{aligned}
&E_{\mu_\rho}\big[(\eta_s(0)-\rho)(\eta_0(0)-\rho)\big] \\
&\quad = \rho(1-\rho)\Big\{E_{\mu_\rho}[\eta_s(0)|\eta_0(0)=1] - E_{\mu_\rho}[\eta_s(0)|\eta_0(0)=0]\Big\} \\
&\quad = \rho(1-\rho)\bar{P}(R_s=0).
\end{aligned}
$$

In the symmetric model, the displacement rate

$$p(z)(1 - \eta''(R_t + z)) + p(-z)\eta''(R_t + z) = p(z)$$

does not depend on the underlying configuration!

–So, the statistics of R_t,
in this case,
is that of a random walk.

In the asymmetric model, however,
formally the drift of the second-class particle is

$$(1-2\rho)\sum zp(z).$$

–Its statistics are more involved.

–In $d=1$, for nearest-neighbor systems, the a.s. law of large numbers,

$$\frac{1}{t}R_t \to \gamma(1-2\rho),$$

holds
Ferrari '92, Rezakhanlou '95, Balazs-Nagy 2017.

Fluctuations of the second-class particle are also of interest.

–In $d = 1$, for nearest-neighbor asymmetric systems, these connect to fluctuations of the current and KPZ class scalings
Ferrari-Spohn 2007, Quastel-Valko 2007.

In particular,

$$\mathrm{Var}(R_t) \sim t^{4/3}$$

(Balazs-Seppalainen 2010).

–In the integrable probability literature, there is much more detailed information for TASEP, etc.
e.g. Ferrrari-Ghoshal-Nejjar 2019.

–In $d = 2$, for $(\uparrow, \rightarrow)$ systems when $\rho = 1/2$,

$$E\left|R_t\right|^2 \sim t(\log t)^{2/3}$$

(Yau 2002).

–In $d \geq 3$, for finite-range systems,

$$E\left|R_t\right|^2 \sim t$$

(Landim-Olla-Varadhan 2004).

Back to occupation time

Recall $f(\eta) = \eta(0) - \rho$.

In the finite-range symmetric model,
we have

$$\mathrm{Var}(A_f(t)) \sim \begin{cases} t^{3/2} & d = 1 \\ t\log(t) & d = 2 \\ t & d \geq 3. \end{cases}$$

In the asymmetric model, when $\rho \neq 1/2$,
the 'velocity' $\gamma(1-2\rho)$ of R_t does not vanish.

In this case, it turns out

$$\mathrm{Var}(A_f(t)) \sim t$$

However, when $\rho = 1/2$, based on the order of $E\left|R_t\right|^2$, one conjectures that the transition probability

$$\bar{P}(R_t = 0) \sim \begin{cases} t^{-2/3} & d = 1 \\ t^{-1/2}(\log t)^{-1/3} & d = 2 \\ t^{-1/2} & d \geq 3. \end{cases}$$

Accordingly,

$$\mathrm{Var}(A_f(t)) \sim t \int_0^t \bar{P}(R_s = 0)ds$$

should match the order of $Var(R_t)$.

–This has been proved for all (ρ, d), except in $d \leq 2$ for asymmetric Exclusion when $\rho = 1/2$.

Here, only superdiffusive bounds have been shown:

$$Var(A_f(t)) \geq \begin{cases} t\log\log(t) & d=2 \\ t^{5/4} & d=1. \end{cases}$$

–See background reference
Bernardin-Goncalves-SS 2015.

Limit distributions

Given the variance orders, what are the distributional limits of the scaled, centered occupation time?

Consider symmetric finite-range Exclusion.
Here, $\chi(\rho)$ is proportional to $\sqrt{\rho(1-\rho)}$.

–When $d = 1$, we have

$$\frac{1}{N^{3/4}} \int_0^{Nt} \left(\eta_s(0) - \rho\right) ds \Rightarrow \chi(\rho) fBM_{3/4}(t)$$

–When $d = 2$,

$$\frac{1}{\sqrt{N \log N}} \int_0^{Nt} \left(\eta_s(0) - \rho\right) ds \Rightarrow \chi(\rho) BM(t).$$

–When $d \geq 3$,

$$\frac{1}{\sqrt{N}} \int_0^{Nt} \big(\eta_s(0) - \rho\big) ds \Rightarrow \chi(\rho) BM(t).$$

Connection to fluctuation fields

We sketch an argument for the $fBM_{3/4}$ limit.

One may approximate in $d = 1$ that

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} E_{\mu_\rho}\Big|\frac{1}{N^{3/2}}\int_0^{N^2 t}\Big\{\big(\eta_s(0)-\rho\big) - \frac{1}{\sqrt{N}}\sum_x i_\epsilon(x/N)\big(\eta_{N^2 s}(x)-\rho\big)\Big\}ds\Big|^2 = 0$$

by a ‘local’ BG principle.

Then, the occupation time

$$\frac{1}{(N^2)^{3/4}} \int_0^{N^2 t} (\eta_s(0) - \rho) ds \sim \int_0^t \frac{\sqrt{N}}{N} \sum_x i_\epsilon(x/N)\big(\eta_{N^2 s}(x) - \rho\big) ds$$
$$= \int_0^t Y_s^N(i_\epsilon) ds.$$

where we recall the fluctuation field

$$Y_t^N\big(i_\epsilon\big) = \frac{1}{\sqrt{N}} \sum_x i_\epsilon(x/N)\big(\eta_{N^2 t}(x) - \rho\big).$$

Since, in the N-limit,

$$\lim_N Y_t^N\big(i_\epsilon(\cdot)\big) = Y_t(i_\epsilon(\cdot)) \stackrel{d}{=} \frac{1}{\sqrt{c}} Y_{c^2 t}\big(i_\epsilon(\cdot/c)\big)$$

we conclude

$$Z_t^\epsilon := \int_0^t Y_s(i_\epsilon) ds$$

satisfies in the ϵ-limit

$$\lim_\epsilon Z_t^\epsilon = Z_t \stackrel{d}{=} \frac{1}{c^{3/4}} Z_{ct}.$$

But, as Y_s^N limits to a Gaussian field,

conclude the N, ϵ-limit Z_t

is a continuous Gaussian process with stationary increments.

Then, Z_t is a $fBM_{3/4}(t)$.

–See Goncalves-Jara 2013; see also SS-Xu '95, SS 2000

Kipnis-Varadhan CLT

Let η_t be a reversible Markov processes, starting in an invariant measure μ.

Theorem (Kipnis-Varadhan '87). Suppose

$$\lim_{t\uparrow\infty} \frac{1}{t}\mathrm{Var}(A_f(t)) = \sigma_f^2 < \infty.$$

Then,

$$\frac{1}{\sqrt{N}} A_f(Nt) = \frac{1}{\sqrt{N}} \int_0^{Nt} f(\eta_s)ds \Rightarrow \sigma_f BM(t).$$

Hence, in $d \geq 3$,
for symmetric nearest-neighbor Exclusion,
starting in μ_ρ, we verify

$$\frac{1}{\sqrt{N}} \int_0^{Nt} \big(\eta_s(0) - \rho\big) ds \Rightarrow \chi(\rho) BM(t).$$

A useful inequality

Let η_t be a Markov process,
starting from an invariant measure μ.

<u>Lemma</u> We have

$$\mathrm{Var}\big(A_f(t)\big) \leq Ct\ E_\mu\big[f(1/t - L)^{-1} f\big].$$

Here,

$$u = (\lambda - L)^{-1} f = \int_0^\infty e^{-\lambda t} P_t dt$$

is the solution of the resolvent equation

$$\lambda u - Lu = f$$

–The quantity

$$E_\mu\big[f(\lambda - L)^{-1}\big]$$

is an 'H_{-1}' norm, which has variational formulas that can be estimated..

–Both BG and 'local' BG can be proved using this lemma, and estimating the H_{-1} norm for the associated functions.

References

There are several good books on interacting particle systems:

- De Masi-Presutti: Mathematical methods for hydrodynamic limits 1991
- Kipnis-Landim: Scaling limits of interacting particle systems 1999
- Liggett: Interacting particle systems 1985
- Spohn: Large scale dynamics of interacting particles 1991

Thank you!