MULTIPLE RANGE OF RANDOM WALK UP TO THE TIME OF EXIT FROM AN INTERVAL

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Abstract. Consider the p-multiple range $R_N^{(p)}$ which counts the number of points visited exactly $p \geq 1$ times by a one-dimensional simple symmetric random walk starting at $[\alpha N]$, for $\alpha \in (0,1)$, up to the time of exit from $D_N = \{0, 1, \ldots, N\}$. We show that $R_N^{(p)} / \log(N)$ converges weakly to the law of an exponential random variable with mean $1/2$. Moreover, we show, by the method of moments, that the collection of scaled multiple ranges $\{R_N^{(p)} / \log(N) : p \geq 1\}$ in the limit is totally correlated.

1. Introduction and results

Classically, up to time $n \geq 0$, the range $\mathcal{R}_n$ and the multiple range $\mathcal{R}_n^{(p)}$ are the number of sites visited, and number of sites visited exactly $p \geq 1$ times by a simple symmetric random walk on $\mathbb{Z}^d$ starting at the origin. The statistics of $\mathcal{R}_n$ and $\mathcal{R}_n^{(p)}$ have been continually studied, going back at least to the 1951 paper of Dvoretzky and Erdös [7]. For instance, recent developments include [1], [2], and references therein. Less is known however of the statistics of the range and multiple range when the random walk is subject to constraints.

Recently, some works have considered the statistics of the range $R_N = \mathcal{R}_{\tau_N}$ and multiple range $R_N^{(p)} = \mathcal{R}_{\tau_N}^{(p)}$ of simple symmetric random walk up to the (random) time of exit τ_N from scaled domains $D_N \subset \mathbb{Z}^d$. Such a setting is natural in the applied context of 'trapping phenomena'; see [4]. More theoretically, R_N and $R_N^{(p)}$ inform on the fine structure of the 'extremal' sojourn of the random walk, when it reaches a 'boundary'.

The purpose of this note is to determine the scaled limits of the joint statistics of $\{R_N^{(p)} : p \geq 1\}$ for simple symmetric random walk in $d = 1$; see Theorems 1.1 and 1.2. That the trajectory, with respect to $R_N^{(p)}$, must exit the interval D_N at time τ_N suggests that there may be a number of exactly 'few' visited sites perhaps near the exit point. Indeed, we show that $\log(N)$ is the order of the count $R_N^{(p)}$ and that $R_N^{(p)} / \log(N)$ converges weakly to an exponential random variable with mean $1/2$. Moreover, we show that the joint distributions of $\{R_N^{(p)} / \log(N) : p \geq 1\}$ are totally correlated in the limit. We comment that the '$\log(N)$' scaling of $R_N^{(p)}$, the exponential limit, and the total correlation of the scaled joint distribution of multiple range counts are novel and were not anticipated. In particular, since $R_N^{(p)}$ and $R_N^{(q)}$ for $p \neq q$ counts sites in disjoint multiple range sets, their total correlation when scaled in the limit is not obvious.

In comparison, in the classical one-dimensional $\mathcal{R}_n^{(p)}$ random walk multiple range, as the random walk returns continually to the origin, one might feel that there is only a few, random in number, exactly p-visited points at large times n. Indeed, see Lemma 5.1 where we calculate

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$\sup_n \mathbb{E}_0[\mathcal{R}_n^{(1)}] < \infty$, and see also [9] which considers unscaled limits of the multiple range $\mathcal{R}_{2n}^{(p)}$ constrained to return to the origin at time $2n$.

Previously, [3] studied the range R_N of simple symmetric random walk in $d = 1$ starting from $[\alpha N] \in D_N = [0, 1, \ldots, N]$ for $\alpha \in (0, 1)$, showing convergence to an explicit distribution $R_N/N \Rightarrow F_\alpha$ depending on α.

In $d \geq 2$, [5] studied the range R_N and multiple range $R_N^{(p)}$ of simple symmetric random walk starting at $[\alpha N] \in D_N = ND$ up to the time of exit τ_N from D_N for bounded sets $D \subset \mathbb{R}^d$. In $d = 2$, $R_N/(N^2/\log(N))$ and $R_N^{(p)}/(N^2/\log(N))$ converge to $\pi\tau_{\alpha,D}$ and $2\pi^2\tau_{\alpha,D}$. Whereas in $d \geq 3$, R_N/N^2 and $R_N^{(p)}/N^2$ converge to $(d/2)(1-p_{0,d})$ and $(d/2)(1-p_{0,d})^2 p_{0,d}^{p-1}\tau_{\alpha,D}$. Here, $\tau_{\alpha,D}$ is the exit time of Brownian motion from the domain D starting at $\alpha \in D$.

One can understand the scalings $a(N) = N^2/\log(N)$ in $d = 2$ and $a(N) = N^2$ in $d \geq 3$ with respect to R_N and $R_N^{(p)}$ given that (1) τ_N/N^2 converges weakly to $d\tau_{\alpha,D}$, and (2) the a.s. limits of the classical range and multiple ranges $\mathcal{R}_n/a(\sqrt{n})$ to nontrivial constants [7], [8], [11], [12]. Similarly, the scaling $a(N) = N$ in $d = 1$ for the range R_N is consistent as the classical $\mathcal{R}_n/a(\sqrt{n})$ converges weakly to the span of a Brownian motion up to time 1 [10].

In terms of the literature, the limits of $\{R_N^{(p)}/\log(N) : p \geq 1\}$ in $d = 1$ in Theorems 1.1 and 1.2 complete a discussion of the general problem.

We now introduce basic notation and state our results formally. Let $\{X_n : n \in \mathbb{N}\}$ be a one-dimensional simple symmetric random walk on $\mathbb{Z}$, that is,

$$\mathbb{P}[X_{n+1} = x \pm 1 | X_n = x] = \frac{1}{2}.$$

Let $T_x = T_x^{(1)} = \inf\{n \geq 0 : X_n = x\}$ be the first time the point x is visited, and $T_x^{(p)} = \inf\{n > T_x^{(p-1)} : X_n = x\}$ be the time that the point x is visited exactly $p > 1$ times. Let $\tau_N = \min\{T_0, T_N\}$ be the time of exit from $D_N = [0, 1, \ldots, N]$. We will set the starting point of the random walk as $X_0 = [\alpha N]$ where $0 < \alpha < 1$ and $N \in \mathbb{N}$ is a scaling factor.

By translation-invariance, the model is the same as when the random walk starts at the origin and τ_N is the exit time from the interval $[-[\alpha N], N - [\alpha N]]$. In this way, all the random variables $\{R_N^{(p)} : p \geq 1, N \geq 1\}$ may be viewed to be all on the same probability space.

Let Z be an exponential random variable with mean $1/2$. Note that the moment $\mathbb{E}\left[Z^j\right] = j!/2^j$ for $j \geq 1$.

Theorem 1.1. *Let $1 \leq p_1 < \ldots < p_n$ for $n \geq 1$. We have the weak convergence of the joint distribution,*

$$\lim_{N\to\infty} \left(\frac{R_N^{(p_1)}}{\log(N)}, \frac{R_N^{(p_2)}}{\log(N)}, \ldots, \frac{R_N^{(p_n)}}{\log(N)} \right) = (Z, Z, \ldots, Z).$$

From consideration of moments (cf. for instance Section 2.3.e in [6]), Theorem 1.1 follows directly from the following result.

Theorem 1.2. *For $n \geq 1$, let $0 \leq j_1, \ldots, j_n$, $1 \leq p_1 < \cdots < p_n$, and $J_n = \sum_{i=1}^n j_i$. Define also the joint scaled moment,*

$$\mu^N_{j_1,\ldots,j_n}(p_1, \ldots, p_n) = \frac{1}{\log^{J_n}(N)} E_{[\alpha N]} \left[\prod_{i=1}^n \left(R_N^{(p_i)} \right)^{j_i} \right].$$

Then,

$$\lim_{N\to\infty} \mu^N_{j_1,\ldots,j_n}(p_1, \ldots, p_n) = \frac{J_n!}{2^{J_n}} = E\left[Z^{J_n}\right].$$

1.1. **Remarks.** We make a few comments about the main theorems and their proofs.

1. As a consequence of Theorem 1.2, the scaled multiple ranges are indistinguishable in the limit and completely correlated: For $\epsilon > 0$,

$$\lim_{N\to\infty} P_{[\alpha N]}\Big(\sum_{1\le i,k\le n} \Big| \frac{R_N^{(p_i)}}{\log N} - \frac{R_N^{(p_k)}}{\log N} \Big| > \epsilon \Big) = 0.$$

Moreover, unlike for the range R_N in $d = 1$ as mentioned above, the scaled limit in Theorem 1.1 does not depend on $\alpha \in (0, 1)$, that is the distance to the boundaries, as long as the distance is at least of order N.

2. Unlike for the range R_N, there is no 'monotonicity' property to exploit for the multiple range $R_N^{(p)}$ in that a point $x \in D_N$ may be visited exactly p times up to time τ_N but visited more times up to time τ_{N+1} and so may not be present in the count $R_{N+1}^{(p)}$. One may view $R_N^{(p)}$ in terms of local times of the random walk. However, since $p < \infty$, the order and limits of $R_N^{(p)}$ reflect finer structure than would be seen in scaling of the local times where p grows with N. Here, the method of proof, taking advantage of the geometry of the $d = 1$ setting, relies on moment calculations via gambler's ruin estimates and the Markov property.

3. As a complement, we mention that [3] and [5] consider the problem for simple asymmetric random walks with right/left jump probabilities $\ell < r$. Given $\tau_N/N \to (1-\alpha)/(1-p_0)$ a.s., the convergences in probability $R_N/N \to 1-\alpha$ and $R_N^{(p)}/N \to (1-p_0)p_0^{p-1}(1-\alpha)$ follow from that of $\mathcal{R}_n/n \to 1-\alpha$ and $\mathcal{R}_n^{(p)}/n \to (1-p_0)^2 p_0^{p-1}(1-\alpha)$ (cf. [11]), where $p_0 = 2(1-r) < 1$ is the probability a random walk starting at the origin returns.

1.2. **Outline of the proofs of Theorem 1.2.** We first prove the theorem when $n = 1$ and $p_1 = 1$ in Section 2 via an induction argument. Then, in Section 3, we prove the theorem when $n = 1$ and $p_1 > 1$, using some of the calculations in Section 2 and the $p_1 = 1$ result as the base case in another induction. In Section 4, we complete the proof of the full result, using schemes in Section 3.

2. Moments of $R_N^{(1)}$: Proof of Theorem 1.2 for $n = 1$, $p_1 = p$

Let $A_x = A_x^{(1)} = \{T_x < \tau_N, T_x^{(2)} > \tau_N\}$ be the event that the point x is visited exactly once before exit from D_N. Since $R_N^{(1)}$ counts the number of points visited exactly once, we can write $R_N^{(1)} = \sum_x 1_{A_x}$. Therefore, for $j \ge 1$,

$$\left(R_N^{(1)}\right)^j = j! \sum_{x_1<x_2<\cdots<x_j} 1_{A_{x_1}} 1_{A_{x_2}} \dots 1_{A_{x_j}} + \sum_{\substack{x_1,x_2,\dots,x_j \\ \text{not distinct}}} 1_{A_{x_1}} \dots 1_{A_j} = B_1 + B_2,$$

where the summations are over all possible $x_1, \dots, x_j \in D_N$. Note also that B_2 is a finite linear combinations of $(R_N^{(1)})^k$ for $1 \le k \le j-1$.

Our goal in this section is to show Theorem 1.2 when $n = 1$ and $p_1 = 1$: Namely,

$$\mathbb{E}_{[\alpha N]}\left[\left(R_N^{(1)}\right)^j\right] = \sum_{x_1<\cdots<x_j} \mathbb{P}_{[\alpha N]}\left(\bigcap_{i=1}^{j} A_{x_i}\right) + \mathbb{E}_{[\alpha N]}[B_2] \tag{1}$$

$$= \frac{j!}{2^j}\log^j(N) + o(\log^j(N)).$$

We will also assume that $[\alpha N] \notin \{x_1, \dots, x_j\}$ in the sum in (1), as the j terms when $[\alpha N] \in \{x_1, \dots, x_j\}$ can be put into the $o(\log^j(N))$ term.

2.1. **Visitation schedule and four cases.** Given an ordering of the points $x_1 < \cdots < x_j$, let $\mathcal{V} = \mathcal{V}^{(1)} = \{y_i\}_{i=0}^{j+1} \subset \{[\alpha N], 0, x_1, \dots, x_j, N\}$ be an ordered list where $y_0 = [\alpha N]$, $y_{j+1} \in \{0, N\}$ and each $x_1, \dots, x_j$ appears once. By the one-dimensional geometry, there are only four possible 'visiting schedules' $\mathcal{V}$ such that a random walk visits in order the points in $\mathcal{V}$ up to the time of exit from D_N and

$$P_{\mathcal{V}} = \mathbb{P}_{[\alpha N]}\left(T_{y_1} < T_{y_2} < \cdots < T_{y_j} < \min\{\tau_N, \min_{i \le j} T^{(2)}_{y_i}\}\right) > 0.$$

Namely (see Figure 1),

Case 1 $\mathcal{V}_1 = ([\alpha N], x_j, x_{j-1}, \dots, x_1, 0)$ with $x_j < [\alpha N] < N$
Case 2 $\mathcal{V}_2 = ([\alpha N], x_j, x_{j-1}, \dots, x_1, 0)$ with $x_{j-1} < [\alpha N] < x_j < N$
Case 3 $\mathcal{V}_3 = ([\alpha N], x_1, x_2, \dots, x_j, N)$ with $0 < x_1 < [\alpha N] < x_2$
Case 4 $\mathcal{V}_4 = ([\alpha N], x_1, x_2, \dots, x_j, N)$ with $0 < [\alpha N] < x_1$.

Then, for $0 < x_1 < \cdots < x_j < N$, we have

$$\mathbb{P}_{[\alpha N]}\left(\cap_{i=1}^j A_{x_i}\right) = \sum_{k=1}^4 \mathbb{P}_{[\alpha N]}\left(T_{y_1} < \cdots < T_{y_{j+1}} < \min_{1 \le i \le n}\{T^{(2)}_{x_i}\}\right) = \sum_{k=1}^4 P_{\mathcal{V}_k}.$$

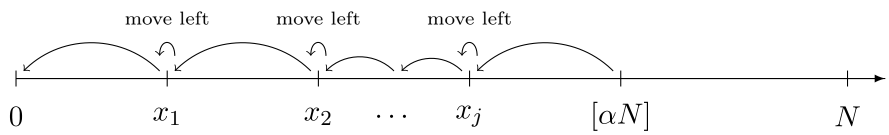

FIGURE 1. Case 1 visitation schedule when $p = 1$

2.2. **Sequential path decomposition.** Recall the gambler's ruin identities for $a < b < c$:

$$\mathbb{P}_b(T_a < T_c) = \frac{c-b}{c-a} \text{ and } \mathbb{P}_b(T_c < T_a) = \frac{b-a}{c-a}.$$

We now describe the sequential decomposition of the probability of paths following $\mathcal{V}_1$. Such a random walk starts at $[\alpha N]$, then visits x_j before visiting N, with chance given by the exit probability $\mathbb{P}_{[\alpha N]}(\{T_{x_j} < T_N\})$. From x_j, the random walk must move left to the point $x_j - 1$, with 'fair' chance $\mathbb{P}_{x_j}(\{T_{x_j - 1} = T_{x_j} + 1\}) = 1/2$, and then visits next x_{j-1} before visiting x_j with chance $\mathbb{P}_{x_j - 1}(\{T_{x_{j-1}} < T_{x_j}\})$. This sequential process continues until the random walk exit at $y_{j+1} = 0$. Thus, by the Markov property and gambler's ruin probabilities,

$$\begin{aligned} P_{\mathcal{V}_1} &= \frac{1}{2^j}\mathbb{P}_{[\alpha N]}(\{T_{x_j} < T_N\})\mathbb{P}_{x_1 - 1}(\{T_0 < T_{x_1}\})\prod_{i=2}^j \mathbb{P}_{x_i - 1}(\{T_{x_{i-1}} < T_{x_i}\}) \\ &= \frac{1}{2^j}\frac{N - [\alpha N]}{N - x_j}\cdot\frac{1}{x_j - x_{j-1}}\cdot\frac{1}{x_{j-1} - x_{j-2}}\cdots\frac{1}{x_2 - x_1}\frac{1}{x_1}. \end{aligned}$$

Let us write

$$\mathbb{E}_{[\alpha N]}[B_1] = j!\sum_{i=1}^4 \sum_{x_1 < \cdots < x_j} P_{\mathcal{V}_i} = \frac{j!}{2^j}\left\{S_j^{(1)} + S_j^{(2)} + S_j^{(3)} + S_j^{(4)}\right\} \tag{2}$$

where

$$S_j^{(1)} = \sum_{x_1<x_2<\cdots<x_j<[\alpha N]} \frac{N-[\alpha N]}{N-x_j}\cdot\frac{1}{x_j-x_{j-1}}\cdot\frac{1}{x_{j-1}-x_{j-2}}\cdots\frac{1}{x_2-x_1}\cdot\frac{1}{x_1} \tag{3}$$

$$S_j^{(2)} = \sum_{x_1<x_2<\cdots<[\alpha N]<x_j} \frac{[\alpha N]-x_{j-1}}{x_j-x_{j-1}}\cdot\frac{1}{x_j-x_{j-1}}\cdot\frac{1}{x_{j-1}-x_{j-2}}\cdots\frac{1}{x_2-x_1}\cdot\frac{1}{x_1}$$

$$S_j^{(3)} = \sum_{x_1<[\alpha N]<x_2<\cdots<x_j} \frac{x_2-[\alpha N]}{x_2-x_1}\cdot\frac{1}{x_2-x_1}\cdot\frac{1}{x_3-x_2}\cdots\frac{1}{x_j-x_{j-1}}\frac{1}{N-x_j}$$

$$S_j^{(4)} = \sum_{[\alpha N]<x_1<x_2<\cdots<x_j} \frac{[\alpha N]}{N-x_j}\cdot\frac{1}{x_j-x_{j-1}}\cdot\frac{1}{x_{j-1}-x_{j-2}}\cdots\frac{1}{x_2-x_1}\cdot\frac{1}{x_1}.$$

We observe that there are relations between Cases 1 and 4 and also between Cases 3 and 4 via the mapping $x\mapsto N-x$. In particular, the order estimates for Cases 1 and 2 equal those of Cases 3 and 4, multiplied by $\alpha/(1-\alpha)$, that is the prefactor α is changed to $1-\alpha$.

2.2.1. *A reduction by partial fractions.* To help reduce the sums, we will invoke the partial fraction decomposition

$$\frac{1}{N-x_i}\cdot\frac{1}{x_i-x_{i-1}} = \frac{1}{N-x_{i-1}}\left[\frac{1}{N-x_i}+\frac{1}{x_i-x_{i-1}}\right]. \tag{4}$$

When $x_i \le [\alpha N]$, we have directly that $\sum_{x_i=x_{i-1}+1}^{[\alpha N]-j-1+i}\frac{1}{N-x_i} = O(1)$, while by the Euler-Maclaurin formula to estimate sums in terms of integrals, we have for $x_{i-1}<x_i$ that

$$\sum_{x_i=[\alpha N]+1}^{N-1}\frac{1}{(x_i-x_{i-1})^2} = \left(\frac{1}{[\alpha N]-x_{j-1}+1}-\frac{1}{N-x_{j-1}-1}\right)+O(1). \tag{5}$$

Similarly, by the Euler-Macluarin formula we have

$$\sum_{x_i=x_{i-1}+A}^{[\alpha N]-j-1+i}\frac{1}{x_i-x_{i-1}} = \log([\alpha N]-x_i-j-i-1)-\log(A)+O(1)\le \log N+O(1). \tag{6}$$

2.3. **Bounding Cases 2 and 3.** These cases involve a 'backtrack' in that from $[\alpha N]$ the walk goes to the end of the sequence, and then must return over the same ground covered, and so their probability will be smaller than in Cases 1 and 4.

Indeed, consider the inner sum in $S_j^{(2)}$. Using the Euler-Maclaurin formula (5) and $x_{j-1}\le [\alpha N]-1$, we get

$$\begin{aligned}\sum_{x_j=[\alpha N]+1}^{N-1}\frac{[\alpha N]-x_{j-1}}{(x_j-x_{j-1})^2} &= ([\alpha N]-x_{j-1})\left(\frac{1}{[\alpha N]-x_{j-1}+1}-\frac{1}{N-x_{j-1}-1}\right)+O(1)\\ &= ([\alpha N]-x_{j-1})\left(\frac{N-[\alpha N]-2}{([\alpha N]-x_{j-1}+1)(N-x_{j-1}-1)}\right)+O(1)=O(1).\end{aligned}$$

Then, we can write $S_j^{(2)}$ as

$$O(1)\cdot\sum_{x_1=1}^{[\alpha N]-j+1}\sum_{x_2=x_1+1}^{[\alpha N]-j+2}\cdots\sum_{x_{j-1}=x_{j-2}+1}^{[\alpha N]-1}\frac{1}{x_{j-1}-x_{j-2}}\cdot\frac{1}{x_{j-2}-x_{j-3}}\cdots\frac{1}{x_2-x_1}\frac{1}{x_1}.$$

Using (6) repeatedly, we can see that the above summation is bounded and

$$S_j^{(2)} = O\left(\log^{j-1}(N)\right). \tag{7}$$

Furthermore, from symmetric notions we can conclude that $S_j^{(3)}$ in Case 3 has the same order $O(\log^{j-1}(N) = o(\log^j(N))$ as in Case 2.

2.4. **Estimating Cases 1 and 2.** We claim that the summation in Case 1 satisfies

$$S_j = S_j^{(1)} = (1-\alpha)\log^j(N) + o(\log^j(N)). \tag{8}$$

By symmetry relations, we would then have $S_j^{(4)} = \alpha \log^j(N) + o(\log^j(N))$. We proceed by induction and iterative estimation to prove (8).

2.4.1. *Base Case.* Let us first show the result for S_1. We use the partial fraction decomposition (4) on (3) to get

$$\begin{aligned} S_1 &= \sum_{x_1=1}^{[\alpha N]-j} \frac{N-[\alpha N]}{N-x_1}\cdot\frac{1}{x_1} \\ &= (N-[\alpha N])\sum_{x_1=1}^{[\alpha N]-j}\frac{1}{N}\cdot\frac{1}{N-x_1} + (N-[\alpha N])\sum_{x_1=1}^{[\alpha N]-j}\frac{1}{N}\cdot\frac{1}{x_1} = \sigma_{(1,1)} + \sigma_{(2,1)} \end{aligned}$$

From inspection (cf. remark after (4)), it follows that $\sigma_{(1,1)} = O(1)$.

On the other hand, using (6) we get $\sigma_{(2,1)} = \left(1 - \frac{[\alpha N]}{N}\right)[\log([\alpha N]-j) + O(1)] = (1-\alpha)\log(N) + O(1)$, and therefore the base induction step holds: $S_1 = (1-\alpha)\log(N) + O(1)$.

2.4.2. *Induction.* Let $j \geq 2$. We have, by partial fractions (4) applied to the inner sum of S_j, that $S_j = \sigma_{(j,1)} + \sigma_{(j,2)}$ where

$$\sigma_{(j,1)} = \sum_{x_1=1}^{[\alpha N]-j}\ \sum_{x_2=x_1+1}^{[\alpha N]-j+1}\cdots\sum_{x_j=x_{j-1}+1}^{[\alpha N]-1} \frac{N-[\alpha N]}{N-x_j}\cdot\frac{1}{N-x_{j-1}}\cdot\frac{1}{x_{j-1}-x_{j-2}}\cdots\frac{1}{x_2-x_1}\cdot\frac{1}{x_1}$$

and

$$\sigma_{(j,2)} = \sum_{x_1=1}^{[\alpha N]-j}\ \sum_{x_2=x_1+1}^{[\alpha N]-j+1}\cdots\sum_{x_j=x_{j-1}+1}^{[\alpha N]-1} \frac{N-[\alpha N]}{x_j-x_{j-1}}\cdot\frac{1}{N-x_{j-1}}\cdot\frac{1}{x_{j-1}-x_{j-2}}\cdots\frac{1}{x_2-x_1}\cdot\frac{1}{x_1}.$$

Applying the remark after (4), it follows that $\sigma_{(j,1)} = O(S_{j-1}) = o(\log^j(N))$ via our induction hypothesis.

Let us now consider $\sigma_{(j,2)}$. Using (6), we see that

$$\begin{aligned} \sigma_{(j,2)} = &\sum_{x_1=1}^{[\alpha N]-j}\ \sum_{x_2=x_1+1}^{[\alpha N]-j+1}\cdots \\ &\sum_{x_{j-1}=x_{j-2}+1}^{[\alpha N]-2} \frac{N-[\alpha N]}{N-x_{j-1}}\cdot\frac{1}{x_{j-1}-x_{j-2}}\cdots\frac{1}{x_2-x_1}\cdot\frac{1}{x_1}\log([\alpha N]-x_{j-1}-1) + O(S_{j-1}). \end{aligned} \tag{9}$$

To evaluate S_j further we ‘sandwich’ it via upper and lower estimates.

2.4.3. *Upper estimate of S_j.* As $\log([\alpha N]-x_{j-1}-1) \le \log(N)$, we have $\sigma_{(j,2)} \le \log(N)\cdot S_{j-1}$. Then, with the bound $\sigma_{(j,1)} = O(S_{j-1})$ and induction, we have

$$S_j = \sigma_{j,1}+\sigma_{j,2} \le (\log(N)+O(1))\, S_{j-1} \le (1-\alpha)\log(N)^j + o(\log(N)^j). \tag{10}$$

2.4.4. *Lower estimate of S_j.* In the sum S_j, we now limit the summands $\{x_i\}$ away each other and the starting position $[\alpha N]$ to get a lower bound. For $\epsilon > 0$ and $i \ge 2$, consider

$$x_{i-1} + \log(N) \le x_i \le [\alpha N] - (j-i+1)\lfloor N^{1-\epsilon}\rfloor.$$

Denote by $S_j^* \le S_j$ the truncated sum

$$S_j^* = \sum_{x_1=\log(N)}^{[\alpha N]-j\lfloor N^{1-\epsilon}\rfloor} \sum_{x_2=x_1+\log(N)}^{[\alpha N]-(j-1)\lfloor N^{1-\epsilon}\rfloor} \cdots \sum_{x_j=x_{j-1}+\log(N)}^{[\alpha N]-\lfloor N^{1-\epsilon}\rfloor} \frac{N-[\alpha N]}{N-x_j}\cdot\frac{1}{x_j-x_{j-1}}\cdots\frac{1}{x_2-x_1}\cdot\frac{1}{x_1}.$$

As in Section 2.4.1, the base case $S_1^* = (1-\alpha)\log(N) + o(\log(N))$.

For $j \ge 2$, analogous to the decomposition of $S_j = \sigma_{(j,1)} + \sigma_{(j,2)}$ via partial fractions in Section 2.4.2, write $S_j^* = \sigma^*_{(j,1)} + \sigma^*_{(j,2)}$ where $\sigma^*_{(j,1)} \le \sigma_{(j,1)} = o(\log^j(N))$ and

$$\begin{aligned}\sigma^*_{(j,2)} = &\sum_{x_1=\log(N)}^{[\alpha N]-j\lfloor N^{1-\epsilon}\rfloor} \sum_{x_2=x_1+\log(N)}^{[\alpha N]-(j-1)\lfloor N^{1-\epsilon}\rfloor} \cdots \sum_{x_{j-1}=x_{j-2}+\log(N)}^{[\alpha N]-2\lfloor N^{1-\epsilon}\rfloor}\\ &\frac{N-[\alpha N]}{N-x_{j-1}}\cdot\frac{1}{x_{j-1}-x_{j-2}}\cdots\frac{1}{x_2-x_1}\cdot\frac{1}{x_1}\\ &\cdot\left[\log([\alpha N]-\lfloor N^{1-\epsilon}\rfloor - x_{j-1}) - \log(\log(N)) + O(S_{j-1})\right].\end{aligned}$$

Observe that $\log([\alpha N]-\lfloor N^{1-\epsilon}\rfloor - x_{j-1})$ is minimized at the largest value of x_{j-1}, which is $[\alpha N]-2\lfloor N^{1-\epsilon}\rfloor$. Thus,

$$\log([\alpha N]-\lfloor N^{1-\epsilon}\rfloor - x_{j-1}) \ge \log(\lfloor N^{1-\epsilon}\rfloor) \ge (1-\epsilon)\log(N-1)$$

and therefore $S_j \ge S_j^* \ge (1-\epsilon)\log(N)S^*_{j-1} + O(\log(\log(N)) + S^*_{j-1}) + o(\log^j(N))$. By iteration of this process and from the base case analysis for S_1^*, we have

$$S_j \ge S_j^* \ge (1-\alpha)(1-\epsilon)^{j-1}\log(N)^j(1+o(1)). \tag{11}$$

2.5. **Conclusion of the proof of Theorem 1.2 when $n=1$ and $p_1=1$.** Given that $\epsilon > 0$ is arbitrary, we may combine the upper estimate in (10) and lower estimate in (11) to complete the induction step (8) to determine that $S_j^{(1)} = (1-\alpha)\log^j(N) + o(\log^j(N))$ and $S_j^{(2)} = \alpha\log^j(N) + o(\log^j(N))$. Hence, adding to $S_j^{(2)}, S_j^{(3)} = o(\log^j(N))$ (cf. (7)), and multiplying by $j!/2^j$ as in (2), we recover (1) and complete the proof of Theorem 1.2 when $n=1$, $p_1=1$.

3. Moments of $R_N^{(p)}$: Proof of Theorem 1.2 when $n=1$ and $p_1=p>1$

Fix $p>1$ and let $A_x^{(p)} = \{T_x^{(p)} < \tau_N, T_x^{(p+1)} > \tau_N\}$ be the event that the point x is visited exactly p times before the time of exit τ_N. Then, analogous to the computation when $p=1$,

$$\left(R_N^{(p)}\right)^j = \sum_{x_1,x_2,\ldots,x_j} 1_{A^{(p)}_{x_1}} 1_{A^{(p)}_{x_2}} \cdots 1_{A^{(p)}_{x_j}} = j!B_1^{(p)} + B_2^{(p)},$$

where $B_1^{(p)} = \sum_{x_1<x_2<\cdots<x_j} 1_{A_{x_1}^{(p)}} 1_{A_{x_2}^{(p)}} \ldots 1_{A_{x_j}^{(p)}}$ and $B_2^{(p)}$ is the sum over non-distinct summands, equal to a linear combination of $R_N^{(k)}$ for $1 \leq k \leq p-1$. Our goal will be to show Theorem 1.2 when $n = 1$ and $p_1 = p > 1$: Namely,

$$\mathbb{E}_{[\alpha N]}\left[R_N^{(p)}\right] = \sum_{x_1<\cdots<x_j} \mathbb{P}_{[\alpha N]}\left(\bigcap_{i=1}^{j} A_{x_i}^{(p)}\right) + \mathbb{E}_{[\alpha N]}\left[B_2^{(p)}\right] = \frac{j!}{2^j}\log^j(N) + o(\log^j(N)). \tag{12}$$

Again, we may take in the sum that $[\alpha N] \notin \{x_1, \ldots, x_j\}$ as the finite possibilities $[\alpha N] \in \{x_1, \ldots, x_j\}$ can be put in the $o(\log^j(N))$ term.

We now extend our notation for visitation schedules from Section 2.1. With respect to points $x_1 < \cdots < x_j$, a valid visitation schedule $\mathcal{V}^{(p)} = \{y_i\}_{i=0}^{jp+1}$ is a list of $jp+2$ members specifying the order in which the collection of points $\{x_i\}_{i=1}^{j}$ are visited exactly p times before exit. Let $x_0 = [\alpha N]$ and $x_{j+1} \in \{0, N\}$.

Define $\ell : \{0, \ldots, jp+1\} \to \{0, 1, \ldots, j, j+1\}$ as the map, which characterizes $\mathcal{V}^{(p)}$, where $y_i = x_{\ell(i)}$. Being valid implies $y_0 = x_0$, $y_1 = x_{\ell(1)}$ is one of possibly two nearest adjacent values to x_0 in $\{x_i\}_{i=1}^{j}$, $y_2 = x_{\ell(2)}$ is one of $x_{\ell(1)}, x_{\ell(1)-1}$ or $x_{\ell(1)+1}$, and so on to $y_{jp} = x_{\ell(k)}$ which must be the nearest adjacent point to either 0 or N, equal to the last value y_{jp+1}.

As a simple example, we remark when $j = 1$ that there are only two visitation schedules

$$\mathcal{V}^{(p)} = ([\alpha N], x_1, x_1, \ldots, x_1, 0 \text{ or } N)$$

where x_1 is repeated p times. On the other hand, when $j > 1$, the visitation schedules are less restricted than when $p = 1$. Trajectories no longer need to visit the $\{x_i\}$ points monotonically from either the left or the right, nor is the starting position $[\alpha N]$ restricted.

By a crude bound, as each y_i takes at most $j+2$ values, the number of visitation schedules $\mathcal{V}^{(p)}$ is at most $(j+2)^{jp+2}$. Notice also in any visitation schedule $\mathcal{V}^{(p)}$ that one can extract an increasing subsequence $\{y_{i_k}\}$ where either $y_{i_1} = x_1$, $y_{i_2} = x_2$, …, $y_{i_j} = x_j, y_{i_{j+1}} = N$ or $y_{i_1} = x_j$, …, $y_{i_j} = x_1, y_{i_{j+1}} = 0$.

For $x_1 < \cdots < x_j$, using the Markov property, write $\mathbb{P}_{[\alpha N]}\left(\bigcap_{i=1}^{j} A_{x_i}^{(p)}\right) = \sum_{\mathcal{V}^{(p)}} P_{\mathcal{V}^{(p)}}$ where

$$P_{\mathcal{V}^{(p)}} = \sum_{\mathcal{V}^{(p)}} \prod_{i=0}^{jp+1} \mathbb{P}_{x_{\ell(i)}}\left(T_{x_{\ell(i+1)}} < \min\{\tau_N, T_{x_{\ell(i+1)-1}}, T_{x_{\ell(i+1)+1}}\}\right). \tag{13}$$

Note that $x_{\ell(i+1)}$ may equal $x_{\ell(i)}$, $x_{\ell(i)\pm 1}$ or 0 or N, depending on the schedule, and there may be some redundancy in the event $\left\{T_{x_{\ell(i+1)}} < \min\{\tau_N, T_{x_{\ell(i+1)-1}}, T_{x_{\ell(i+1)+1}}\}\right\}$. Therefore,

$$\mathbb{E}_{[\alpha N]}\left[R_N^{(p)}\right] = j! \sum_{\mathcal{V}^{(p)}} \sum_{x_1<\ldots<x_j} P_{\mathcal{V}^{(p)}} + \mathbb{E}_{[\alpha N]}\left[B_2^{(p)}\right], \tag{14}$$

where $\mathbb{E}_{[\alpha N]}\left[B_2^{(p)}\right]$ is a finite linear combination of $\mathbb{E}_{[\alpha N]}\left[R_N^{(k)}\right]$ for $1 \leq k \leq p-1$.

3.1. **Visitation schedules with backtracking.** For a random walk to traverse between two points x_i and x_{i+1} more than once, that is to 'backtrack', is costly due to an extra gambler's ruin factor in (13). This can happen if the initial point $[\alpha N]$ is in between the $\{x_i\}$ or if a pair (x_i, x_{i+1}) is traversed more than once, as we saw in the analysis of Case 2 in Section 2.3 when $n = 1$ and $p_1 = 1$; see Figure 2.

In particular, suppose $x_i < x_0 = [\alpha N] < x_{i+1}$ a journey $[\alpha N]$ to either x_i or x_{i+1} yields a gambler's ruin factor with $x_{i+1} - x_i$ in the denominator, either $\mathbb{P}_{[\alpha N]}(T_{x_i} < T_{x_{i+1}}) = \frac{x_{i+1}-[\alpha N]}{x_{i+1}-x_i}$

or $\mathbb{P}_{[\alpha N]}(T_{x_i} > T_{x_{i+1}}) = \frac{[\alpha N]-x_i}{x_{i+1}-x_i}$. Then, as the random walk will need to make a backtrack between the two points x_i and x_{i+1} to visit all of the points $p \geq 2$ times, we get an additional gambler's ruin factor given by $(x_{i+1}-x_i)^{-1}$. There may also be more backtracks between x_i and x_{i+1} in the schedule; these result in gambler's ruin factors which we bound by 1.

When summed over x_i, by applying Euler-Macluarin (5) and noting $x_{i+1} > [\alpha N]$, we have

$$\sum_{x_i=x_{i-1}+1}^{[\alpha N]-1} \frac{x_{i+1}-[\alpha N]}{(x_{i+1}-x_i)^2} = (x_{i+1}-[\alpha N])\left[\frac{1}{x_{i+1}-[\alpha N]+1} - \frac{1}{x_{i+1}-x_{i-1}+1}\right] + O(1) \tag{15}$$

$$= (x_{i+1}-[\alpha N])\frac{[\alpha N]-x_{i-1}}{(x_{i+1}-[\alpha N]+1)(x_{i+1}-x_{i-1}+1)} + O(1) = O(1).$$

The other case with factor $([\alpha N]-x_i)/(x_{i+1}-x_i)^2$ is similar when summing over x_{i+1}, resulting in an $O(1)$ sum.

Another way a 'backtrack' occurs is when the random walk goes between x_i and x_{i+1} more than once. In this situation, the gambler's ruin factors are bounded by $(x_{i+1}-x_i)^{-2}$. When summed over either x_i or x_{i+1} the effect is $O(1)$ (cf. (5); see Figure 2).

For visiting schedules $\mathcal{V}^{(p)}$ in such situations, suppose that the increasing subsequence $\{y_{i_k}\}$ starts at x_j and exits at 0, as in Case 1 when $n = 1$ and $p_1 = 1$ (cf. Section 2.1). In the situation when the starting position $y_0 = [\alpha N]$ is between an x_{i-1} and x_i and say $y_1 = x_i$ (the other case $y_1 = x_{i+1}$ is similar), we conclude

$$P_{\mathcal{V}^{(p)}} \leq \mathbb{P}_{[\alpha N]}(T_{x_i} < T_{x_{i-1}})\mathbb{P}_{x_j}(T_{x_{j-1}} < T_N)\left[\prod_{l=1}^{j-2} \mathbb{P}_{x_{j-l}}(T_{x_{j-l-1}} < T_{x_{j-l}})\right]\mathbb{P}_{x_1}(T_0 < T_{x_1}).$$

The sum over $x_1 < x_2 < \cdots < x_j$ of the last display is similar to the expression $S_j^{(2)}$ in Section 2.3, with the difference being that the $O(1)$ sum may be in the middle, rather than at the beginning. Indeed, noting (15) and by virtually the same calculations as in Section 2.3, the sum is of order $O(\log^{j-1}(N))$.

Similarly, in the situation where $\mathcal{V}^{(p)}$ has a backtrack where a pair x_i and x_{i+1} is traversed more than once, or when $\mathcal{V}^{(p)}$ has a backtrack and the increasing subsequence starts at x_1 and ends at N, we obtain $\sum_{x_1<\cdots<x_j} P_{\mathcal{V}^{(p)}} = O(\log^{j-1}(N))$.

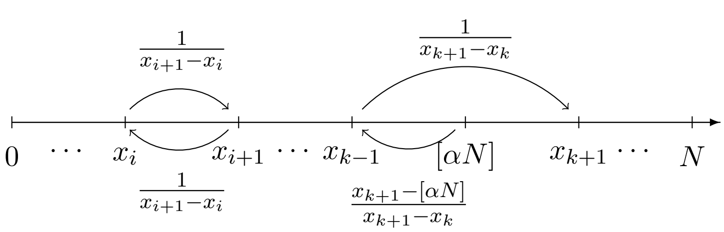

FIGURE 2. Two types of backtracking: when the random walk travels between two points x_i and x_{i+1} at least twice, or when $[\alpha N]$ is in between the points

3.2. **Visiting schedules without 'backtracking'.** When there is no backtracking, we have only two scenarios; see Figure 3.

Case 1 $x_j < [\alpha N] < N$ and $\mathcal{V}^{(p)} = ([\alpha N], x_j, ..., x_{j-1}, \ldots, x_1, \ldots, x_1, 0)$

Case 2 $0 < [\alpha N] < x_1$ and $\mathcal{V}^{(p)} = ([\alpha N], x_1, \ldots, x_1, \ldots, x_j, \ldots, x_j, N)$

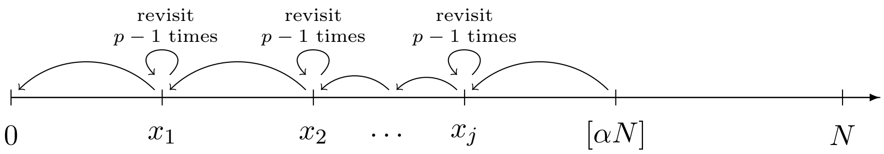

FIGURE 3. Case 1 visitation schedule when $p > 1$

where each value x_i is repeated p times.

For a schedule $\mathcal{V}^{(p)}$ without backtracking, say in Case 1 going from $[\alpha N]$ to x_j repeated p times, and so forth, exiting at 0, we have

$$P_{\mathcal{V}^{(p)}} = \frac{1}{2^j}\frac{N-[\alpha N]}{N-x_j}\cdot\frac{1}{x_j-x_{j-1}}\cdot\frac{1}{x_{j-1}-x_{j-2}}\cdots\frac{1}{x_2-x_1}\frac{1}{x_1}\prod_{i=1}^{j}(\mathbb{P}_{x_i}(I_{x_i}))^{p-1}$$

where $I_{x_i} = \left\{T^R_{x_i} < \min\{\tau_N, \min\{T_{x_k} : k \neq i\}\}\right\}$ is the event of return to x_i before visiting other points or exit. Here, $T^R_x = \inf\{n > 0 : X_n = x\}$ is the return time, in contrast to the visit time T_x.

A similar expression holds in Case 2.

Here, for $2 \le i \le j-1$,

$$\mathbb{P}_{x_i}(I_{x_i}) = \frac{1}{2}\mathbb{P}_{x_i-1}\left(T_{x_i} < T_{x_{i-1}}\right) + \frac{1}{2}\mathbb{P}_{x_i+1}\left(T_{x_i} < T_{x_{i+1}}\right) = 1 - \frac{1}{2}\left[\frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1}-x_i}\right]$$

and $\mathbb{P}_{x_1}(I_{x_1}) = 1 - \frac{1}{2}\left[\frac{1}{x_1} + \frac{1}{x_2-x_1}\right]$ and $\mathbb{P}_{x_j}(I_{x_j}) = 1 - \frac{1}{2}\left[\frac{1}{x_j-x_{j-1}} + \frac{1}{N-x_j}\right]$. Of course, $\mathbb{P}_{x_\cdot}(I_{x_\cdot}) \le 1$ and when the points $\{x_i\}$ are separated from each other and the boundaries $0, N$ by $\log(N)$, we have $\mathbb{P}_{x_\cdot}(I_{x_\cdot}) \ge 1 + O(\log^{-1}(N))$.

The corresponding sum $\sum_{x_1<\cdots<x_j} P_{\mathcal{V}^{(p)}}$ is analogous to that for $S^{(1)}_j$ when $p = 1$, except for the factors $\mathbb{P}_{x_\cdot}(I_{x_\cdot})$. Indeed, an upper bound is $S^{(1)}_j$ On the other hand, the sum restricted to $\{x_i\}$ separated from each other by $\log(N)$, as in the definition of S^*_j in Section 2.4.4, is a lower bound. We have

$$S^*_j(1 + O(\log^{-1}(N)) \le \sum_{x_1<\cdots<x_j} P_{\mathcal{V}^{(p)}} \le S^{(1)}_j.$$

Hence, following the steps and conclusion in Section 2.5,

$$\sum_{x_1<\cdots<x_j} P_{\mathcal{V}^{(p)}} = \frac{1-\alpha}{2^j}\log^j(N) + o(\log^j(N)). \tag{16}$$

The sum with respect to Case 2 will evaluate analogously as $(\alpha/2^j)\log^j(N) + o(\log^j(N))$.

3.3. **Conclusion and proof of Theorem 1.2 when $n = 1$ and $p_1 = p > 1$.** Consider induction on p. The base case $p = 1$ has already been shown in Section 2.5. In the equation (14) for the moment, the term $\mathbb{E}\left[B^{(p)}_2\right] = O(\log^{j-1}(N))$ as it is a finite linear combination of lower order moments. For the other term in (14), as there are only a finite number of schedules, we need only consider schedules without backtracking by the $o(\log^j(N))$ estimates for those which backtrack in Section 3.1.

On the other hand, for the two schedules without backtracking, we have the estimate (16) in Case 1 and the complementary one in Case 2, which when added together, and multiplied by $j!$, yield the desired dominant contribution $(j!/2^j)\log^j(N)$, verifying (12), completing the proof of Theorem 1.2 when $n=1$ and $p_1=p>1$.

4. Mixed moments and proof of Theorem 1.2

Recall the event $A_x^{(p)}=\{T_x^{(p)}<\tau_N<T_x^{(p+1)}\}$ that the point x is visited exactly p times. For $1\le p_1<\cdots<p_n$ and $0\le j_1,\ldots,j_n$, recall $J_n=\sum_{i=1}^n j_i$ and consider the (unscaled) joint moment, $m^N_{j_1,\ldots,j_n}(p_1,\ldots,p_n)=\log^{-J_n}(N)\mu^N_{j_1,\ldots,j_n}(p_1,\ldots,p_n)$,

$$\begin{aligned} m^N_{j_1,\ldots,j_n}(p_1,\ldots,p_n) &= \mathbb{E}_{[\alpha N]}\Big[\big(R_N^{p_1)}\big)^{j_1}\cdots\big(R_N^{(p_n)}\big)^{j_n}\Big] \\ &= \mathbb{E}_{[\alpha N]}\Big[\sum 1_{A^{(p_1)}_{x^1_1}}\cdots 1_{A^{(p_1)}_{x^1_{j_1}}}\cdots 1_{A^{(p_n)}_{x^n_1}}\cdots 1_{A^{(p_n)}_{x^n_{j_n}}}\Big], \end{aligned}$$

where the sum is over $x_1^1,\ldots,x^1_{j_1},\ldots,x^n_1,\ldots,x^n_{j_n}$.

We may apply virtually the same analysis as for the moments of $R_N^{(p)}$ in Section 3. Listing the indices to be visited in order as $z_1,\ldots,z_{J_n}$, the dominant terms arise when they are distinct in D_N and not equal to $[\alpha N]$. With the same calculations as in Section 3, only visiting schedules without backtracking among the $\{z_i\}$ contribute to the dominant order of the moment, of which there are only two. Analogous to Case 1 and 2 in Section 3.2, the only difference is that when the random walk first visits the point $x^r_\cdot$, it must return $r-1$ times before moving on, rather than exactly $p-1$ times. However, when the elements $\{x^r_\cdot : r=1,\ldots,n\}$ are separated by $\log(N)$, then the return probabilities equal $1+O(\log^{-1}(N))$ as in Section 3.2.

Hence, the dominant order of the moment $m^N_{j_1,\ldots,j_n}(p_1,\ldots,p_n)$ is the same as for the J_nth moment of $R_N^{(1)}$. In particular, the scaled joint moment $\mu^N_{j_1,\ldots,j_n}(p_1,\ldots,p_n)=\frac{J_n!}{2^{J_n}}(1+o(1))$, from which Theorem 1.2 follows.

5. Expected order of $\mathcal{R}_n^{(1)}$

To be brief, we concentrate on $\mathcal{R}_n^{(1)}$, although a similar, more involved argument should show that $\mathbb{E}_0\big[\mathcal{R}_n^{(p)}\big]=O(1)$.

Lemma 5.1. *We have* $\sup_{n\ge1}\mathbb{E}_0\big[\mathcal{R}_n^{(1)}\big]<\infty$.

Proof. Write $\mathcal{R}_n^{(1)}=\sum_x 1_{\{T_x\le n<T_x^{(2)}\}}$ and $\mathbb{E}\big[\mathcal{R}_n^{(1)}\big]=\sum_x\mathbb{P}_0\Big(T_x\le n<T_x^{(2)}\Big)$. By translation invariance, recall $\mathbb{P}_x(T_x>2\ell)=\mathbb{P}_0(T_0>2\ell)=\mathbb{P}_0(X_{2\ell}=0)$ (cf. equation (3.4), p. 197 in [6]) and so, by local central limit theorem (cf. Theorem 5.2, p. 130 in [6]), that $\sup_x\sup_\ell\sqrt{r+1}\mathbb{P}_x(T_x>r)<\infty$. Then,

$$\begin{aligned} \mathbb{E}\big[\mathcal{R}_n^{(1)}\big] &= \sum_x\sum_{k=1}^n \mathbb{P}_0\left(T_x=k\right)\mathbb{P}_x\left(T_x>n-k\right) \\ &= \sum_x\sum_{k=0}^n \mathbb{P}_0\left(T_x=k\right)\mathbb{P}_0\left(T_0>n-k\right)\le C\sum_{k=0}^n\frac{1}{\sqrt{n-k+1}}\sum_x\mathbb{P}_0\left(T_x=k\right). \end{aligned}$$

By the reflection principle, $\mathbb{P}_0(T_x<\ell)=2\mathbb{P}_0(X_\ell>|x|)$ for $x\neq0$. Hence, for $k\ge1$, as the increment $X_{k+1}-X_k$ takes values ±1 with equal probability, and by local central limit theorem again,

$$\begin{aligned}\sum_{x\neq 0}\mathbb{P}_0(T_x=k) &= \sum_{x\neq 0}\left(\mathbb{P}_0(T_x<k+1)-\mathbb{P}_0(T_x<k)\right)\\ &= 2\sum_{x\neq 0}\left(\mathbb{P}_0(X_{k+1}>|x|)-\mathbb{P}_0(X_k>|x|)\right) = \sum_{x\neq 0}\left(\mathbb{P}_0(X_k=|x|)-\mathbb{P}_0(X_k=|x|+1)\right)\\ &= \mathbb{P}_0(X_k=-1)+\mathbb{P}_0(X_k=1)\le \frac{C}{\sqrt{k+1}}.\end{aligned}$$

Note the boundary cases: $\sum_x \mathbb{P}_0(T_x=k)=\mathbb{P}_0(T_0=0)=1$ when $k=0$, and $\mathbb{P}_0(T_x=k)\le \mathbb{P}_0(X_k=0)\le C/\sqrt{k+1}$ when $x=0$ by local central limit theorem. Then, we have $\sup_n \mathbb{E}\big[\mathcal{R}_n^{(1)}\big]\le C\sum_{k=0}^n \frac{1}{\sqrt{n-k+1}}\frac{1}{\sqrt{k+1}}\le C\int_0^1 \frac{1}{\sqrt{u(1-u)}}du<\infty$ as desired. $\square$

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