Power broadening of a diffusion resonance

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We derive an expression in terms of cylinder functions for the shape of a nonlinear resonance in a two-level system with a rapidly decaying level. We show that when the natural linewidth is negligible, the square of the total width is the sum of squares of the power and diffusion widths. The traditional variational approximation yields a correct value for the full width at half maximum, but distorts the line profile. We derive a formula for the absorbed power as a function of the incident wave intensity for comparable power and diffusion broadening. The formula is found to be valid for a power width that is small or large compared to the diffusion width, and in a new intermediate domain where homogeneous saturation becomes inhomogeneous.
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1. INTRODUCTION

The subject of this paper is resonant absorption (or amplification) of a strong electromagnetic wave by ions of a plasma. Knowledge of the laws of amplification is necessary in calculating ion lasers and in optimizing the characteristics of such lasers (for example, to increase the output power or tuning range of a Raman laser). Measuring the absorption (or amplification) spectrum of waves propagating through a plasma is one of the most important plasma diagnostic techniques.

From a theoretical standpoint, the simplest and most important case (from a practical standpoint) is that of resonant interaction of an electromagnetic wave and a plasma when the wave frequency ω is close to the Bohr frequency $\omega_{21} = (E_2 - E_1)/\hbar$ of the transition between intrinsic states $|1\rangle$ and $|2\rangle$ of an ion. Because of the Doppler effect, the wave interacts most effectively with ions that satisfy the resonance condition $\mathbf{k}\cdot\mathbf{v} = \omega - \omega_{21}$, where $\mathbf{k}$ is the wave vector and $\mathbf{v}$ is the ion velocity. The power absorbed from a weak electromagnetic field in the linear approximation (when changes in the ion state initiated by the wave are ignored) is proportional to the number of particles that interact with the wave. As the frequency varies, the wave resonantly interacts with ions that have a different velocity. The width of the absorption spectrum is determined by the characteristic spread of the velocities v_T of the ions, i.e., by the width of the Maxwellian velocity distribution, and is equal to kv_T. By measuring the linear absorption spectrum we can find the ion temperature.

Much more information about the plasma parameters and the relaxation processes inside an individual ion can be gathered if one employs the methods of nonlinear spectroscopy. An electromagnetic wave interacting with the plasma tends to equalize the populations of the intrinsic states of the ions. Because of the resonant nature of this interaction, the velocity distribution of the populations acquires narrow nonequilibrium structures known as Bennett dips or peaks.[1] The shape of Bennett dips can be found by the probe field technique by measuring the absorption of another electromagnetic wave as a function of frequency. The absorption spectrum of the probe wave that resonantly interacts with the same transition has a nonlinear resonance, a dip in the vicinity of the frequency at which the probe wave resonantly interacts with those ions whose populations have been partially equalized by the first wave. A similar dip emerges in the absorption spectrum of a standing wave consisting of two counterpropagating traveling waves.

The velocity of an ion in a plasma changes with time, since the ion is in the field of the other charged particles comprising the plasma. In contrast to gases, where collisions with other atoms are infrequent but in each such collision the atom dramatically changes velocity, an ion is constantly in a rapidly varying field, which leads to diffusion variations in its velocity.[2] The description of collisions as diffusion in velocity space is also possible for heavy neutral particles in a buffer gas of light particles.

The first theoretical study of the diffusion shape of the Bennett dip in a weak field as applied to ions in a plasma was Ref. 3. It was shown that because of diffusion in velocity with a coefficient D, the dip in level j acquires a width $\sqrt{D/\Gamma_j}$ equal to the characteristic variation of the ion velocity during the lifetime Γ_j^{-1} in level j. If the diffusion width exceeds the natural width, the dip in the distribution over the projection of velocity on the wave vector has a characteristic cusp at the center described by the exponential function $\exp(-|x|)$, where x is the dimensionless deviation of the velocity projection from the center of the dip. Quenching of levels by electrons and other processes described by the relaxation-constant model may add to the natural width of the dip.

In the limit of a weak electromagnetic field, the depth of Bennett dips is proportional to the wave's intensity. But the increase in the depth with wave intensity slows down as the level populations become equalized. In a strong field, the populations at the center of the resonance are essentially equal, and the range of velocities in which the population difference decreases substantially broadens as the field intensity increases.[4] The problem of adding diffusion broadening to power broadening was studied in Ref. 5 by a variational method, and the Lorentzian contour was taken as the trial function for the dip profile. When the diffusion and power widths are comparable, the shape of the dip is some sort of average between a Lorentzian and an exponential. In Ref. 6 the shape of the dip was found in the limit of a small natural width. In the present paper we discuss this problem in greater detail.

The saturation curve, i.e., the dependence of the power absorbed from the field on the intensity of the incident wave, also makes it possible to measure a number of plasma parameters and relaxation constants of the transition. A knowledge of the analogous dependence of the gain is required to calculate the output power of a laser. Diffusion leads to a situation in which some of the ions depart from resonance as their velocities change, while other ions that have not yet absorbed (or emitted) a photon begin to participate in the resonance process. Because of this, the power absorbed from the field (or imparted to it) increases. This dependence in relatively weak fields, where the power width of the Bennett dip is much less than the diffusion width, was derived in Ref. 7 and experimentally corroborated by Apolonsky *et al.*[8] It was shown that in the presence of diffusion the absorption saturation becomes homogeneous and sets in at lower field intensities than in the absence of diffusion. The intensity at which saturations sets in proved to be proportional to the square root of the diffusion coefficient D. The case in which the diffusion and power widths of Bennett dips are comparable remained unexplored.

In Sec. 2 we give the initial equations for the ion density matrix, a classification of the processes of interaction of an ion and the field of electromagnetic waves, and the characteristic values of the main parameters. In Sec. 3 we derive an expression for the shape of the Bennett dip when there is both power broadening and diffusion broadening. In Sec. 4 we calculate the saturation curve, i.e., the power absorbed from a strong field. For this curve we give simple interpolation formulas, which make it possible to forego the assumption that the radiative width must be small. Section 5 contains a qualitative discussion of the results.

2. BASIC EQUATIONS

We normalize the Maxwellian velocity distribution of the ions to unity:

$$W(\mathbf{v})=\frac{1}{(\sqrt{\pi}v_T)^3}\ \exp\left(-\frac{\mathbf{v}^2}{v_T^2}\right),$$

$$v_T=\sqrt{\frac{2T}{m}},\qquad \int d\mathbf{v}W(\mathbf{v})=1,$$

where T is the temperature of the ions in the plasma, and m is the ion mass. We take two excited ion states $|1\rangle$ and $|2\rangle$ with energies E_1 and E_2 ($E_1<E_2$) and examine the resonant interaction of the given two-level system with a traveling electromagnetic wave

$$\mathbf{E}(t,\mathbf{r})=\tfrac{1}{2}(\mathbf{E}_0e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}+\text{c.c.})$$

that is offset from resonance by $\Omega=\omega-\omega_{21}\ll\omega$, where $\omega_{21}=(E_2-E_1)/\hbar$. Assuming a dipole ion–wave interaction, in the resonance approximation we have the following quantum transport equations for the density matrix in the Wigner representation:[9]

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\Gamma_1\right)\rho_1=\nu\hat{\mathscr{V}}\rho_1-2\,\mathrm{Re}(iG^*e^{i(\Omega t-\mathbf{k}\cdot\mathbf{r})}\rho_{21})+q_1W(\mathbf{v})+A_{21}\rho_2, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\Gamma_2\right)\rho_2=\nu\hat{\mathscr{V}}\rho_2+2\,\mathrm{Re}(iG^*e^{i(\Omega t-\mathbf{k}\cdot\mathbf{r})}\rho_{21})+q_2W(\mathbf{v}),$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\Gamma_{12}\right)\rho_{21}=\nu\hat{\mathscr{V}}\rho_{21}+iGe^{-i(\Omega t-\mathbf{k}\cdot\mathbf{r})}(\rho_2-\rho_1),$$

where

$$\hat{\mathscr{V}}=\frac{\partial}{\partial v_\alpha}\Phi_{\alpha\beta}(\mathbf{v})\left(\frac{v_T^2}{2}\frac{\partial}{\partial v_\beta}+v_\beta\right);$$

$\rho_j=\langle j|\hat{\rho}|j\rangle$ and $\rho_{21}=\langle 2|\hat{\rho}|1\rangle$ are the matrix elements of the density matrix; Γ_1, Γ_2, and Γ_{12} are relaxation constants of the states $|1\rangle$ and $|2\rangle$ and of the ‘‘coherence’’ ρ_{21} of these states; $G=\mathbf{E}_0\langle 2|\hat{\mathbf{d}}|1\rangle/2\hbar$, with $\hat{\mathbf{d}}$ the dipole moment operator; $q_jW(\mathbf{v})$ is the excitation function of $|j\rangle$; A_{21} is the Einstein coefficient; the diffusion operator $\hat{\mathscr{V}}$ describes diffusion in velocity space; $\Phi_{\alpha\beta}(\mathbf{v})$ is the diffusion tensor; and ν is the transport collision rate, or the reciprocal of the time it takes a particle to change velocity (due to diffusion) by a quantity of order v_T.

Effects related to the dependence of $\Phi_{\alpha\beta}$ on $\mathbf{v}$ were discussed in Ref. 10. There it was shown that for $\nu\ll\Gamma_j$, i.e., when the diffusion width of Bennett dips along $\mathbf{k}$ is much less than v_T, we can adopt a constant and isotropic diffusion coefficient $\Phi_{\alpha\beta}(\mathbf{v})=\delta_{\alpha\beta}$, introduced into nonlinear spectroscopy by Rautian.[11] If we align the z axis with $\mathbf{k}$, then only the diffusion tensor component $\Phi_{zz}(\mathbf{v})$ averaged over the transverse velocity and taken at the longitudinal velocity corresponding to the center of the dip affects the shape of the Bennett dip. The adopted approximation makes it possible to integrate the system of equations (1) with respect to velocities transverse to $\mathbf{k}$. For the sake of brevity, we denote the longitudinal velocity projection $v_\parallel=\mathbf{k}\cdot\mathbf{v}/k$ by v.

Bearing in mind the various applications to nonlinear spectroscopy, we are interested in the steady-state solution of the system of equations (1):

$$\rho_j=r_j,\quad \rho_{21}=r_{21}e^{-i(\Omega t-\mathbf{k}\cdot\mathbf{r})},$$

$$\Gamma_1r_1=\nu\hat{\mathscr{V}}_1r_1-2\,\mathrm{Re}(iG^*r_{21})+q_1W_1(v)+A_{21}r_2,$$

$$\Gamma_2r_2=\nu\hat{\mathscr{V}}_1r_2+2\,\mathrm{Re}(iG^*r_{21})+q_2W_1(v), \tag{2}$$

$$[\Gamma_{12}-i(\Omega-kv)]r_{21}=\nu\hat{\mathscr{V}}_1r_{21}+iG(r_2-r_1),$$

$$W_1(v)=\frac{1}{\sqrt{\pi}v_T}\exp\left(-\frac{v^2}{v_T^2}\right),\quad \hat{\mathscr{V}}_1=\frac{d}{dv}\left(\frac{v_T^2}{2}\frac{d}{dv}+v\right).$$

The power absorbed from the wave per unit volume is the product of the concentration of ions absorbing a photon per unit time and the energy of the photon:

$$P(\Omega)=2\hbar\omega\int dv\ \mathrm{Re}(iG^*r_{21}). \tag{3}$$

We now present numerical values of the characteristic parameters, which can be divided into plasma parameters and the parameters of a singe ion. The ion temperature T_i characteristic of discharge is roughly 1 eV, and the concentration of ions in the plasma of an argon laser, N_i, is roughly $10^{14}\ \mathrm{cm}^{-3}$. The wavelength of the radiation that is in resonance with the laser transitions of a singly charged ion, λ, is approximately 5×10^{-5} cm. The relaxation constants Γ_j and Γ_{ij} are determined by the rates of relaxation processes and quenching processes in the plasma. Depending on the levels selected, these constants can vary from 10^7 to $10^9\ \mathrm{s}^{-1}$. For Ar II we have $v_T\simeq2\times10^5$ cm/s, $kv_T\simeq2.5\times10^{10}\ \mathrm{s}^{-1}$, and the transport collision rate

$$\nu=\frac{16\sqrt{\pi}e^4N_i}{3m^2v_T^3}L\simeq10^7\ \mathrm{s}^{-1},$$

where e is the electron charge and L is the Coulomb logarithm. The parameters of the gas-discharge plasma have been thoroughly discussed in Ref. 9. The above values of the parameters were used in the numerical calculations. Note that all of the relaxation constants (Γ_j and Γ_{ij}) and the transport collision rate ν are small compared to the Doppler linewidth kv_T.

3. BENNETT DIP IN COMBINED POWER AND DIFFUSION BROADENING

In the absence of diffusion, the shape of the Bennett dip Δr_j is Lorentzian, and the width of the dip is the sum of the homogeneous width Γ_{12} and the power width w_F (Ref. 4):

$$\Delta r_j\propto[\Gamma_{12}^2+w_F^2+(\Omega-kv)^2]^{-1},$$

$$w_F=\sqrt{\frac{2\Gamma_{12}|G|^2}{k^2}\left(\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2}-\frac{A_{21}}{\Gamma_1\Gamma_2}\right)}.$$

If $r_j(v)$ has width w, then in order to estimate the order of magnitude of the individual terms in the equation for r_j we must replace the derivative d/dv by w^{-1}. The collisional term consists of two terms, $(\nu v_T^2/2)d^2r_j/dv^2$ and $\nu(d(vr_j)/dv)$. The first describes velocity diffusion, while the second describes dynamic friction on an ion moving with respect to the entire plasma. The diffusion width w_{jD} of the Bennett dip at level j comes into play because of the equality of Γ_jr_j and the diffusion term, $(\nu v_T^2/2)d^2r_j/dv^2\sim(\nu v_T^2/2w_{jD}^2)r_j$, with the result that $w_{jD}=v_T\sqrt{\nu/2\Gamma_j}$. For moderate concentrations of charge carriers in the plasma, $\nu\ll\Gamma_j$ and $w_{jD}\ll v_T$, and we can neglect terms corresponding to dynamic friction. Due to friction, the variation of the ion velocity during the lifetime Γ_j^{-1} is small in comparison to the diffusion variation of the velocity, w_{jD}, and is of order $v_T\nu/\Gamma_j$.

Below we examine the shape of the Bennett dip in two limits:

1. The upper level is long-lived, and the lower level rapidly decaying ($\Gamma_2\ll\Gamma_1$). This situation leads, for equal rates of excitation of levels, to steady-state population inversion, and is typical of laser transitions of cw ion lasers.[9]

2. The lower level is long-lived, or $\Gamma_1\ll\Gamma_2$. A metastable lower state is used in the absorbing transition of Raman lasers.[8]

In both cases we assume that the diffusion width of the Bennett dip in the short-lived level is much less than the homogeneous or power width. Neglecting diffusion operators in the population equations of this level and for the coherence r_{21} in the system of equations (2), we obtain an ordinary second-order differential equation for the population r_2 of the long-lived level in case 1:

$$\left[1+J\left(1-\frac{A_{21}}{\Gamma_1}\right)\right]r_2=w_{2D}^2\frac{d^2r_2}{dv^2}+r_2^{(0)}+J\left(r_1^{(0)}-\frac{A_{21}}{\Gamma_1}r_2^{(1)}\right), \tag{4}$$

where

$$J=\frac{2\Gamma_{12}|G|^2}{\Gamma_2[\Gamma_{12}^2+2\Gamma_{12}|G|^2/\Gamma_1+(\Omega-kv)^2]},$$

and $r_j^{(0)}$ is the population of level j in the absence of a field. The shape of the Bennett dip is described by the function $y=(r_2-r_2^{(0)})/(r_1^{(0)}-r_2^{(0)})$, which obeys the equation

$$\left(1+\frac{A^2}{W^2+x^2}\right)y=\frac{d^2y}{dx^2}+\frac{A^2}{W^2+x^2}, \tag{5}$$

where

$$x=\frac{\Omega-kv}{kw_{2D}},\quad A=\frac{w_F}{w_{2D}},\quad W=\frac{(\Gamma_{12}^2+2\Gamma_{12}|G|^2/\Gamma_1)^{1/2}}{kw_{2D}}.$$

Note that although $|G|^2$ is present in the expression for W, the latter is not simply the ratio A of the power width to the diffusion width. For instance, in a strong field, $W\simeq(\Gamma_2/\Gamma_1)^{1/2}A\ll A$. The exact solutions of the homogeneous equation can be expressed in terms of spheroidal functions.[12] These functions, however, have been rather poorly studied and no integral representations are known for them.

In case 2, reasoning along similar lines, we have

$$\left[1+J\left(1-\frac{A_{21}}{\Gamma_2}\right)\right]r_1=w_{1D}^2\frac{d^2r_1}{dv^2}+r_1^{(0)}+J\left(1-\frac{A_{21}}{\Gamma_2}\right)r_2^{(0)}, \tag{6}$$

where the expression for J can be obtained from its counterpart in case 1 via the interchange $1\leftrightarrow2$. If we interchange the indices in the expressions for x, y, A, and W, we again arrive at Eq. (5).

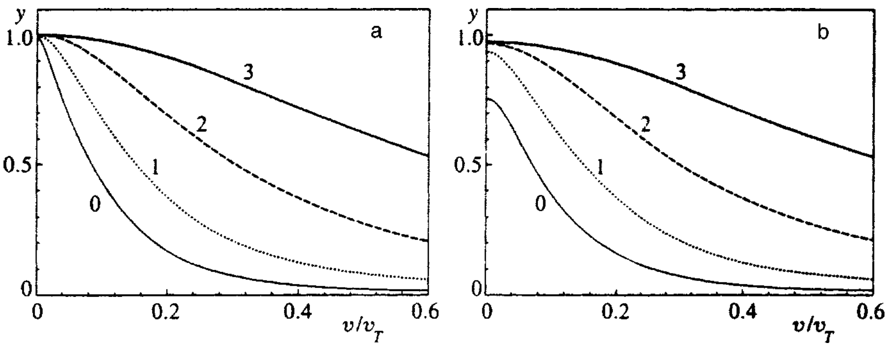

FIG. 1. The Bennett dip $y(x)$: (a) the solution (10), and (b) the results of numerical calculations. $\Gamma_{<}=10^{-3}kv_T$ and $\Gamma_{>}=4\times 10^{-2}kv_T$, where $\Gamma_{<}$ ($\Gamma_{>}$) is the smaller (larger) of the two quantities Γ_1 and Γ_2. $\Gamma_{12}=(\Gamma_1+\Gamma_2)/2$, $\nu=10^{-5}kv_T$, and $\Omega=0$. The curves $n=0,1,2,3$ correspond to the different values of $|G|=10^{n/3-2}kv_T$.

Equation (5) has a simple physical meaning: there is a reservoir of particles and a set of states that differ in the value of the parameter x. In these states, a particle experiences diffusion in x and decay (the unity on the left-hand side of the equation). The external field resonantly interacts with particles, initiating transitions between the states x and the reservoir; the resonance width is W. There is a nonvanishing probability of arrival or departure of a particle in state x, and this probability is $A^2/(W^2+x^2)$.

We now analyze Eq. (5). If either A or W is large ($W \gg 1$ or $A \gg 1$), the width of the Bennett dip exceeds the diffusion width, and we can ignore the derivative d^2y/dx^2 in the equation, so that the Bennett dip becomes Lorentzian:

$$y=\frac{A^2}{W^2+A^2+x^2}. \tag{7}$$

In weak fields, $A \ll W \sim 1$, the dip is a convolution of Lorentzian and exponential contours:

$$y=\frac{1}{2}e^{-|x|} * \frac{A^2}{W^2+x^2} \equiv \frac{A^2}{2}\int dx' \, \frac{e^{-|x-x'|}}{W^2+x'^2}. \tag{8}$$

We now examine the case in which the diffusion width of the Bennett dip in the long-lived level exceeds the homogeneous width, $W \ll 1$. Replacing $1/(W^2+x^2)$ by $\pi\delta(x)/W$, we obtain

$$y=\frac{d^2y}{dx^2}+\frac{\pi A^2}{W}\delta(x)[1-y(0)],$$

whose solution is

$$y(x)=\frac{A^2}{2W/\pi+A^2}e^{-|x|}. \tag{9}$$

This solution is valid for weak fields, $A \ll 1$. As the field intensity grows, the depth of the dip, $y(0)$, approaches unity, and the accuracy of the approximation based on this substitution decreases. In very strong fields, $A \sim 1$, the shape of the Bennett dip differs from the exponential shape $e^{-|x|}$.

Let us construct a solution of Eq. (5) for large values of x ($x \gg W$). We will then attempt to correlate the solution's behavior with the condition of solvability for small values of $x \sim W$. When $x \gg W$, we can ignore W^2 in the denominators of (5). Then the solution $y(x)$ can be expressed in terms of cylinder functions:

$$y(x)=A^2\sqrt{i|x|}\,[CK_\alpha(|x|)-S_{-3/2,\alpha}(i|x|)], \tag{10}$$

where $\alpha=\sqrt{A^2+1/4}$, $K_\alpha(z)$ and $S_{\mu,\alpha}(z)$ are the modified Bessel function of the second kind and Lommel function,[13] and C is a constant that can be found by examining the behavior of the solution (10) at small x.

When $x \sim W$, we can ignore the unity on the left-hand side of Eq. (5), $A \gg W$. The resulting inhomogeneous equation has the trivial solution $y \equiv 1$. The solution of the homogeneous equation can be expressed in terms of hypergeometric series ${}_2F_1$ (Ref. 14). However, when $A^2 \gg W$, the constant C can be found from the condition that the solution (10) must be regular at $x=0$:

$$C=\frac{e^{-i\alpha\pi/2}}{2^{3/2}\pi}\Gamma\left(\frac{-1/2-\alpha}{2}\right)\Gamma\left(\frac{-1/2+\alpha}{2}\right)\times\cos\left[\frac{\pi}{2}\left(\frac{3}{2}+\alpha\right)\right].$$

The solution (10) can then be written in terms of the modified Bessel function $I_\alpha(z)$ and the generalized hypergeometric series ${}_1F_2$ (Ref. 14),

$$y(x)={}_1F_2\left(1;\frac{3/2+a}{2},\frac{3/2-a}{2};\frac{x^2}{4}\right)-\sqrt{\frac{|x|}{2}}\Gamma\left(\frac{3/2+\alpha}{2}\right)\Gamma\left(\frac{3/2-\alpha}{2}\right)I_\alpha(|x|),$$

which implies that $y(0)=1$. The solution (10) contains the Lorentzian and exponential contours as limiting cases:

$$y=\begin{cases} e^{-|x|}, & A \ll 1,\\ \dfrac{1}{1+(x/A)^2}, & A \gg 1. \end{cases} \tag{11}$$

In weak fields, $A^2 \sim W \ll 1$, the Bennett dip is described by the exponential contour (9) with $y(0)\neq 1$.

Figure 1 depicts examples of Bennett dips: (a) the solution (10), and (b) the numerical solutions of four coupled diffusion equations for the elements of the density matrix in (2). The range of possible velocities in (2) is infinite, so for numerical calculations the limits of this interval were taken at $v=\pm 4v_T$, and the values of the density matrix at these limits were chosen according to the asymptotic behavior of the analytic Lorentzian. Figure 1 clearly shows the transition from an exponential contour to a Lorentzian contour (from curve $n=0$ to curve $n=3$). The curves $n=1,2,3$ in Fig. 1a almost coincide with the curves $n=1,2,3$ in Fig. 1b; the dif-

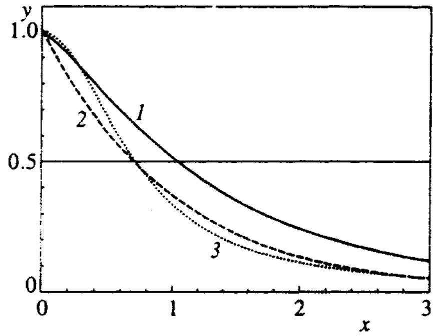

FIG. 2. The shape of the dip $y(x)$ given by formula (10) at $A=\sqrt{1/2}$ (curve *1*), the diffusion contour $\exp(-|x|)$ (curve *2*), and the Lorentzian $1/(1+(x/A)^2)$ (curve *3*).

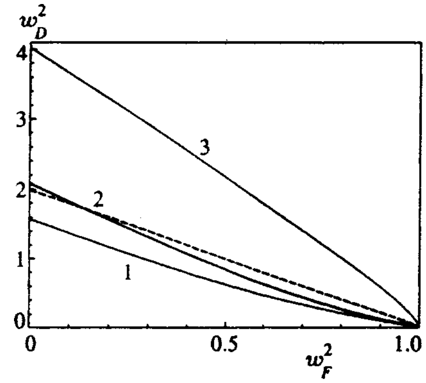

FIG. 3. Level curves for $v_{n/4}$, $n=1,2,3$, corresponding to $v_{n/4}=\sqrt{4/n-1}$. The dashed curve depicts the approximation (12) for the curve $n=2$.

ference between Figs. 1a and 1b at the center of the Bennett dip in a weak field (curve $n=0$) can be explained by the fact that only in the numerical calculations have we allowed for the finite natural width $W\neq 0$.

The type of solution $y(x)$ given by (10) depends only on the parameter $A=w_F/w_D$, where w_D is the diffusion width of the Bennett dip in the long-lived level ($w_D=w_{2D}$ in case 1 and $w_D=w_{1D}$ in case 2). Thus, the halfwidth v_L of the dip over a fraction L $(0<L<1)$ of the total height of the dip ($v_L=w_D x_L$, where $y(x_L)=L$) is a homogeneous function of w_D and w_F of order unity. This means that if w_D and w_F are increased p-fold ($p>0$), then so is v_L. In the (w_F^2,w_D^2) plane, the various level curves of v_L with the same value of L but different values of v_L, can be mapped into one another via a scale transformation centered at $(0,0)$. For instance, the halfwidth at half maximum is

$$v_{1/2}=\begin{cases} w_F\left[1+\left(\dfrac{w_D}{2w_F}\right)^2\right], & w_D\ll w_F, \\ w_D \ln 2, & w_D\gg w_F, \end{cases}$$

and can be approximated by the interpolation formula

$$v_{1/2}\simeq\sqrt{w_F+w_D^2/2}=w_D\sqrt{A^2+1/2}. \qquad (12)$$

Note that the simplicity of (12) derives from the approximate equality $\sqrt{2}\cdot\ln 2=0.9803\ldots\simeq 1$ (if $w_F=0$, then (12) yields $v_{1/2}=w_D/\sqrt{2}\simeq w_D \ln 2$). At $A=1/\sqrt{2}$ the width of the Bennett dip exceeds both the power width and the diffusion width by a factor of almost 1.5. Such a contour is depicted in Fig. 2.

Babin *et al.*[5] used a Lorentzian function for an approximate solution. The amplitude and width were found by a variational method. The width of the approximating Lorentzian satisfies the quartic equation

$$v_{1/2}^4-(w^2+w_F^2+w_D^2/2)v_{1/2}^2-2ww_D^2v_{1/2}-3w^2w_D^2/2=0, \qquad (13)$$

where $w=w_D W$ (in Ref. 5 w was taken to be $w_H=\Gamma_{12}/k$, which is justified if the field is not too strong). Its solution when $w_D,w_F\gg w$ is given by (12), i.e., the halfwidths at half maximum of the solution of (13) and the approximating Lorentzian in the variational method are essentially identical. The level curves for the solution of Eq. (13) in the (w_F^2,w_D^2) plane are segments of straight lines, which become parallel for $w_D,w_F\gg w$, and can thus be mapped into one another by a scale transformation centered at the origin.

The values of $v_{n/4}$, $n=1,2,3$, in Fig. 3 are selected in such a way that the curves $n=1,2,3$ coincide if the dip is a pure Lorentzian. Thus, the distance between curves is the measure of the deviation of the shape from a Lorentzian. The deviation is greater at the center of the dip than in the wings (curve $n=3$ deviates from curve $n=2$ more than does curve $n=1$) and if the power broadening is smaller than the diffusion broadening. When the power width is large, $w_F\gg w_D$, the curves merge, so that saturation effects mask diffusion broadening.

The modified Bessel function $K_\nu(z)$ for half-integer $\nu=n+1/2$, with $n=0,\pm1,\pm2,\ldots$, can be expressed in terms of elementary functions. However, the solution (10) of the inhomogeneous equation reduces to elementary functions only if $\alpha=2n+1/2$, $n=0,1,\ldots$. For instance, at $\alpha=1/2$ the contour is exponential (see (11)), while at $\alpha=5/2$ the shape of the dip is given by

$$y(x)=\frac{6-2e^{-|x|}(3+3|x|+x^2)}{x^2},$$

which differs from a Lorentzian of width $\sqrt{13/2}$ (see Eq. (13) and Ref. 5) by 8%.

We now attempt to find the spectrum of a probe field. The shape of the Bennett dip can be found by measuring the absorption spectrum for a probe wave that is in resonance with the same transition as the strong wave. To determine the spectrum of the probe wave, we must augment the system of equations (1), so that it reflects the interaction with the probe wave. In the linear approximation in the probe field intensity, the probe field induces the coherence ρ_{21} and population corrections, which oscillate in time with frequencies Ω_μ and $\Omega_\mu-\Omega$, where Ω_μ is the probe wave offset from resonance. The corrections to the density matrix induced by the probe field are proportional to the population difference when the longitudinal velocity is in resonance with the probe field. If we assume that the rates of excitation of levels 1 and 2 are the same, the population of the long-lived level will be much higher than that of the rapidly decaying. Due to field splitting of levels, the resonant velocity is a nonlinear function of

Ω_μ. This leads to a situation in which the absorption (amplification) spectrum for the probe wave acquires dips whose shape does not follow that of a Bennett dip. The profile is distorted by the nonlinear dependence of the resonant velocity on Ω_μ, that is, by power broadening.

4. SATURATION CURVE

First and foremost, we show that the power absorbed from the field can be expressed solely in terms of the area of the Bennett dip, $\int_{-\infty}^{\infty} dx\ y(x)$. For case 2, substituting $2\,\mathrm{Re}(iG^* r_{21})$ from the first equation in (2), we obtain

$$P(\Omega)=2\hbar\omega\int dv\ \mathrm{Re}(iG^* r_{21})$$

$$=\hbar\omega\Gamma_1\int dv\left[w_{1D}^2\frac{d^2 r_1}{dv^2}-r_1+r_1^{(0)}+\frac{A_{21}}{\Gamma_1}(r_2-r_2^{(0)})\right]. \tag{14}$$

Expressing r_1 and r_2 in terms of y, $r_1^{(0)}$, and $r_2^{(0)}$, we have

$$P(\Omega)=\hbar\omega\Gamma_1\int dv(r_1^{(0)}-r_2^{(0)})\times\left[y-\frac{d^2y}{dx^2}+\frac{A_{21}}{\Gamma_1}J(1-y)\right].$$

Since the width of the Bennett dip is much less than v_T, we can take $r_1^{(0)}-r_2^{(0)}$ outside the integral sign and evaluate it at $v=\Omega/k$. Noting that $J(1-y)=(y-d^2y/dx^2)\times(1-A_{21}/\Gamma_2)^{-1}$, we obtain

$$P(\Omega)=\frac{\hbar\omega\Gamma_1 w_{1D}(N_1-N_2)}{\sqrt{\pi}v_T(1-A_{21}/\Gamma_2)}\times\exp\left[-\left(\frac{\Omega}{kv_T}\right)^2\right]\int_{-\infty}^{\infty}dx\ y(x), \tag{15}$$

where $N_j=\int dv\ r_j^{(0)}$ is the initial concentration of ions in level j. Similarly, for case 1 we have

$$P(\Omega)=\frac{\hbar\omega\Gamma_2 w_{2D}(N_1-N_2)}{\sqrt{\pi}v_T}\exp\left[-\left(\frac{\Omega}{kv_T}\right)^2\right]\int_{-\infty}^{\infty}dx\ y(x). \tag{16}$$

Thus, studying the intensity dependence of the power absorbed from the field reduces to studying the behavior of the integral $\int_{-\infty}^{\infty}dx\ y(x)$, where y is the solution of Eq. (5).

This fact has a simple qualitative explanation. In the steady-state situation, the area of the Bennett dip in level j, i.e., $\int dv(r_j-r_j^{(0)})$, is the ratio of the concentration of ions that absorb (or emit) one field photon per unit time to the population relaxation rate for the given level Γ_j. The power absorbed from the field can be expressed in terms of the area of the Bennett dip if the rate of particle ejection from the level due to collisions is independent of the translational velocity of the particles (e.g., collisions have no effect on the number of particles in the level).

Consider the Fourier transform of Eq. (5):

$$\left(W^2+\frac{A^2}{1+t^2}\right)Y=\frac{d^2Y}{dt^2}+2\pi A^2\delta(t), \tag{17}$$

where

$$Y(t)=(1+t^2)\int_{-\infty}^{\infty}dx\ e^{itx}y(x).$$

The entire dependence of the absorbed power $P(\Omega)$ on the wave's intensity is contained in the amplitude of the zeroth-order harmonic $Y(0)=\int_{-\infty}^{\infty}dx\ y(x)$.

When $W\gg A$, we can neglect the term $A^2/(1+t^2)$ on the left-hand side of Eq. (17), with the result that $Y(t)=\pi A^2\exp(-W|t|)/W$. When $W\gg 1$ or $A\gg 1$, the function $Y(t)$ has a width $(W^2+A^2)^{-1/2}\ll 1$. We then can assume that t is small, $t\ll 1$, and

$$Y(t)=\frac{\pi A^2}{\sqrt{W^2+A^2}}\exp(-\sqrt{W^2+A^2}|t|).$$

In a different limiting case, $W\ll\min\{A,1\}$, we solve Eq. (17) in two ranges t, namely $t\gg 1$ and $t\ll A/W$. When $A\gg W$, these ranges overlap, and when $1\ll t\ll A/W$, both asymptotes are applicable, which makes it possible to match them, with the result that we obtain an approximation for $Y(t)$ that is equally suitable for all values of t.

For $t\gg 1$ we can neglect the 1 in the denominator $1+t^2$, with the result that Eq. (17) reduces to Bessel's differential equation. This yields

$$Y(t)=C_3\sqrt{t}K_\alpha(Wt), \tag{18}$$

where C_3 is a constant determined by matching (18) to the solution $Y(t)$ for small t.

When $t\ll A/W$, we can neglect W^2 on the left-hand side of Eq. (17), whereupon (17) reduces to the hypergeometric equation.

$$Y(t)=C_{1\pm}(i+t)_2F_1\left(\frac{1}{2}+\alpha,\frac{1}{2}-\alpha;2;\frac{i+t}{2i}\right)+C_{2\pm}(i-t)_2F_1\left(\frac{1}{2}+\alpha,\frac{1}{2}-\alpha;2;\frac{i-t}{2i}\right); \tag{19}$$

for $t>0$ the constants C_1 and C_2 carry a plus sign, while for $t<0$ they carry a minus sign. Here ${}_2F_1(a,b;c;z)$ is the hypergeometric series (see Ref. 14).

What remains to be done is to find the constants $C_{1\pm}$, $C_{2\pm}$, and C_3. Constraints on the behavior of $Y(t)$ near $t=0$ impose two constraints on the values of the constants; another emerges when we match the asymptotic solutions (18) and (19), and finally, the fact that $y(x)$ is real leads to two more.

Integrating (17) locally near $t=0$, we find that

$$\lim_{t\to+0}Y(t)=\lim_{t\to-0}Y(t),$$

$$\lim_{t\to-0}\frac{dY(t)}{dt}-\lim_{t\to+0}\frac{dY(t)}{dt}=2\pi A^2, \tag{20}$$

$$C_{1+}+C_{2+}=C_{1-}+C_{2-},$$

$$C_{1-}-C_{2-}-C_{1+}+C_{2+}=\frac{2\pi A^2}{{}_2F_1-A^2{}_2F_1^+/4},$$

where

$${}_2F_1={}_2F_1\left(\frac{1}{2}+\alpha,\frac{1}{2}-\alpha;2;\frac{1}{2}\right),$$

$${}_2F_1^+={}_2F_1\left(\frac{3}{2}+\alpha,\frac{3}{2}-\alpha;3;\frac{1}{2}\right).$$

Since $y(x)$ is real, $Y(-t)=Y^*(t)$, so that $C_{m-}=-C_{m+}^*$, with $m=1,2$.

When $1\ll t\ll A/W$, the solutions (18) and (19) must coincide. Here the argument of the hypergeometric series in (19) is large, $t\gg 1$, and in (18) the argument of the modified Bessel function can be chosen to be small, $Wt\ll 1$. Expanding (18) as a power series in t at $t=0$ (see Ref. 13) and (19) as a power series in t^{-1} at $t=\infty$ (see Ref. 14),

$$Y(t)=\frac{\pi C_3\sqrt{t}}{2\sin\pi\alpha}\left[\frac{(Wt/2)^{-\alpha}}{\Gamma(1-\alpha)}-\frac{(Wt/2)^{\alpha}}{\Gamma(1+\alpha)}\right]+\cdots,$$
$$Y(t)=\exp\left[-\frac{i\pi}{2}\left(\alpha+\frac{1}{2}\right)\right](C_{1+}-ie^{i\pi\alpha}C_{2+})\times\frac{\Gamma(-2\alpha)}{\Gamma(1/2-\alpha)\Gamma(3/2-\alpha)}\left(\frac{t}{2}\right)^{1/2-\alpha}$$
$$+\exp\left[\frac{i\pi}{2}\left(\alpha-\frac{1}{2}\right)\right](C_{1+}-ie^{-i\pi\alpha}C_{2+})\times\frac{\Gamma(2\alpha)}{\Gamma(1/2+\alpha)\Gamma(3/2+\alpha)}\left(\frac{t}{2}\right)^{1/2+\alpha}+\cdots, \tag{21}$$

and equating the coefficients of $t^{1/2\pm\alpha}$, we obtain

$$\frac{C_{1+}-ie^{-i\pi\alpha}C_{2+}}{C_{1+}-ie^{i\pi\alpha}C_{2+}}=\varepsilon e^{-i\pi\alpha},$$
$$\varepsilon=-W^{2\alpha}\frac{\Gamma(-2\alpha)\Gamma(1-\alpha)\Gamma(1/2+\alpha)\Gamma(3/2+\alpha)}{\Gamma(2\alpha)\Gamma(1+\alpha)\Gamma(1/2-\alpha)\Gamma(3/2-\alpha)}. \tag{22}$$

Representing C_{1+} in the form $C_{1+}=\mu C_{2+}$, we obtain

$$\mu=ie^{-i\pi\alpha}\frac{1-\varepsilon e^{i\pi\alpha}}{1-\varepsilon e^{-i\pi\alpha}},\quad |\mu|=1,$$

$$Y(0)=iC_{2+}{}_2F_1=-\frac{\pi A^2{}_2F_1}{2({}_2F_1-A^2{}_2F_1^+/4)}\frac{|1+\mu|^2}{\mathrm{Im}\,\mu}$$
$$=\frac{\pi A^2{}_2F_1}{{}_2F_1-A^2{}_2F_1^+/4}\cot\left(\frac{\pi}{2}\left(\alpha-\frac{1}{2}\right)+\arctan\frac{\varepsilon\sin\pi\alpha}{1-\varepsilon\cos\pi\alpha}\right). \tag{23}$$

Thus, when the power width exceeds the radiative width ($A\gg W$), the power absorbed from the field per unit volume is given by (16) for case 1 and by (15) for case 2, with $\int_{-\infty}^{\infty}dx\ y(x)=Y(0)$ given by (23). The power width is comparable to the diffusion width, $A\sim 1$. The expression (23) is cumbersome, so we start with several limiting cases, after which we suggest a simple but accurate approximation.

4.1. Limiting cases

Suppose that the field is weak ($A\ll 1$). Then $\alpha\simeq 1/2+A^2$, $\varepsilon\simeq W^{2\alpha}$, and the argument of $\cot z$ in (23) is small, with the result that

$$Y(0)\simeq\frac{\pi A^2}{W^{2\alpha}+\pi A^2/2}. \tag{24}$$

A similar result, $Y(0)=\pi A^2/(W+\pi A^2/2)$, which differs from (24) only in the exponent of W in the denominator, can be obtained by replacing $A^2/(W^2+x^2)$ in (5) with $\pi A^2\delta(x)/W$ (Ref. 7). The extent to which the approximation based on this substitution differs from (24) can be estimated by the difference $1-W^{2A^2}$, which for $W\ll A\ll 1$ can be of order unity. The expression (24) describes so-called *homogeneous saturation*, where the diffusion shape of the Bennett dip does not change (it remains exponential), while the depth of the dip, $y(0)$, reaches its maximum value, equal to unity, as the field strength increases.

Now consider strong fields ($A\sim 1$). We can then neglect the correction to the argument of $\cot z$ in (23), which is related to the fact that ε is nonvanishing (see Appendix). Setting $\varepsilon=0$ in (23), we can write the asymptotic formulas for $Y(0)$ for strong and weak diffusion broadening as compared to power broadening (see (11)):

$$Y(0)=\begin{cases}2, & A\ll 1,\\ \pi A(1+1/4\,A^2), & A\gg 1.\end{cases} \tag{25}$$

4.2. Interpolation formulas

For strong fields ($A\sim 1$), $Y(0)$ obeys the interpolation formula

$$Y(0)\simeq\pi\sqrt{(2/\pi)^2+A^2}, \tag{26}$$

which, as numerical calculations show, is valid in the region where $W\ll\min\{A^2,1\}$ with an accuracy no worse than 2%.

Combining (26) and (24), we arrive at an interpolation formula with an applicability range broader than (26):

$$Y(0)\simeq\frac{\pi A^2\sqrt{(2/\pi)^2+A^2}}{(2/\pi)W^{2\alpha}+A^2}. \tag{27}$$

This expression is valid in the region $W\ll 1$ and, in particular, in the case of a weak field, $A^2\sim W$.

Finally, altering (27) somewhat, we obtain at the interpolation formula

$$Y(0)\simeq\frac{\pi A^2\sqrt{(2/\pi)^2+W^2+A^2}}{(2/\pi)W^{2\alpha}/(1+2W^{2\alpha})+W^2+A^2}, \tag{28}$$

which is valid for all A, $W>0$ with an accuracy of about 3%. Examples of saturation curves (which reflect the dependence of the power absorbed from the field on the field intensity) calculated with (28) are depicted in Fig. 4. We see that the smaller the natural width W, the sooner homogeneous saturation sets in (the transition from curve *1* to curves *2* and *3* in Fig. 4b). However, a further increase in A^2 (Fig. 4a) leads to a square-root increase in the absorbed power. This increase is almost unnoticeable on the scale of Fig. 4b.

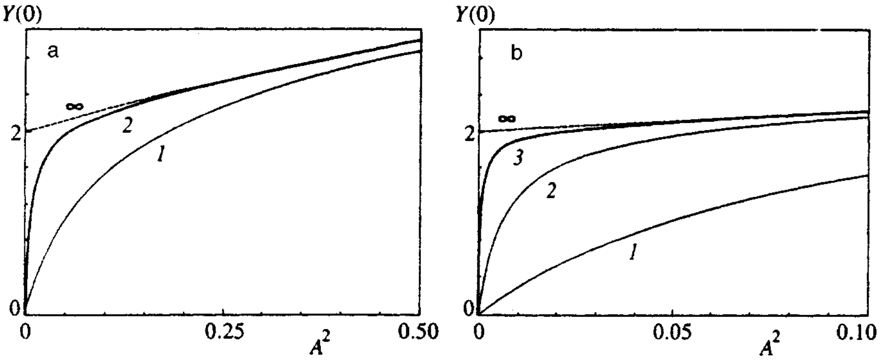

FIG. 4. The absorbed power $Y(0)$ as a function of the field intensity A^2. Curve n corresponds to the value $W=10^{-n}$ (curve ∞ corresponds to $W=0$).

Reverting to the original variables of the problem, we obtain

$$P(\Omega)=\frac{2\sqrt{\pi}\hbar\omega\Gamma_{12}|G|^2(N_1-N_2)}{kv_T}\exp\left[-\left(\frac{\Omega}{kv_T}\right)^2\right]\times\frac{\sqrt{\Gamma_{12}^2+\mathscr{F}^2+(2/\pi)^2\mathscr{D}}}{\Gamma_{12}^2+\mathscr{F}^2+(2/\pi)\mathscr{D}p/(1+2p)},$$

$$\mathscr{F}^2=2\Gamma_{12}|G|^2\left(\frac{1}{\Gamma_1}+\frac{1}{\Gamma_2}-\frac{A_{21}}{\Gamma_1\Gamma_2}\right),\quad \mathscr{D}=\frac{\nu k^2v_T^2}{2\Gamma_<},$$

$$p=\left(\frac{\Gamma_{12}^2+2\Gamma_{12}|G|^2/\Gamma_>}{\mathscr{D}}\right)^{\sqrt{\mathscr{F}^2/\mathscr{D}+1/4}},$$

where $\Gamma_<$ ($\Gamma_>$) is the lesser (greater) of the two quantities Γ_1 and Γ_2. In case 1 we have $\Gamma_<=\Gamma_2$ and $\Gamma_>=\Gamma_1$, while in case 2 we have $\Gamma_<=\Gamma_1$ and $\Gamma_>=\Gamma_2$.

5. DISCUSSION

Two results have been obtained in the previous sections.

1. We derived an expression for the shape of the Bennett dip in a two-level system with a rapidly decaying level, for an arbitrary ratio of the power and diffusion widths of the dip. The expression is valid if the natural width of the dip is small compared to the total width and the width of the Maxwellian distribution exceeds that of the dip. We show that the square of the total dip width at half maximum is the sum of squares of the power and diffusion widths.

2. Under the above conditions, we derived a formula that reflects the dependence of the absorbed power on the intensity of the incident wave. The formula describes the well-known limiting cases of low and high intensity and the smooth transition from homogeneous saturation to inhomogeneous saturation in a new intermediate region. An interpolation formula has been suggested for this dependence. The formula is valid even if the homogeneous width is not small.

5.1. Shape of dip

Equation (5) contains three widths: the diffusion width (equal to unity), the power width A, and the homogeneous width W. The solutions for the shape of the Bennett dip in the collisionless case (both A and W are large compared to unity) and in the case of a weak field (A is small compared to W and unity) were obtained earlier. The only case that remained unexplored was when one of the three widths is small and the other two are comparable (W small). Here we add power broadening to diffusion broadening. The depth $y(0)$ of the Bennett dip for $A^2\gg W$ reaches its maximum $y(0)=1$, while the shape of the dip for $A\ll 1$ is determined by diffusion and is described by an exponential contour. As the field intensity A^2 increases, the dip flattens out. The general case, in which all three widths are comparable, can hardly be solved in terms of hypergeometric functions, which usually contain only two widths. For instance, $J_\nu(z)$ contains two scales: unity (the period of oscillations at large values of z), and ν (the distance from $z=0$ to the origin of oscillations). However, even if we were able to write the solution in some form, it would still be simpler to find it by direct numerical methods in each specific case. Furthermore, when W is of order unity the shape of the dip is close to Lorentzian, so that the variational approximation described in Ref. 5 yields satisfactory results.

5.2. Saturation curve

The expression for the absorbed power obtained in Ref. 7 and formula (9) of the present paper can be applied in two regions: $A^2\ll 1/|\ln W|$, $W\ll 1$, and $W\ll A^2\ll 1$. This provides a correct description of homogeneous saturation for $W\ll 1$. The condition that they can be used to describe homogeneous saturation is the smallness of the parameter $W|\ln W|$, which tends to zero as $W\to 0$. Formula (9) cannot be applied in the case of strong fields ($A\sim 1$), when the power width of the Bennett dip is of the order of the diffusion width. One property of (9) is that for large values of A, the integral $\int dx\, y(x)$ reaches its maximum value, equal to 2. It was unclear how the absorbed power behaves at $A\sim 1$, since it is known that $\int dx y(x)=\pi A$ for $A\gg 1$, i.e., the absorbed power grows with the wave intensity. The expression (23) or the interpolation formulas (27) and (28) describe precisely the case where $A\sim 1$. They can also be applied for weak fields ($A\ll 1$). These expressions cannot be applied when the width of one of the levels is small compared to that of the other, or when the width of the Bennett dip is small compared to the width v_T of the Maxwellian distribution. If the diffusion width of the Bennett dip is less than v_T, saturation associated with the fact that the power width becomes equal to v_T (and hence the square-root increase in the absorbed power, $\int dx\, y(x)=\pi\sqrt{A^2}$, with the field intensity A^2 becoming limited to the extent that the absorbed power finally becomes a constant) can be described without taking diffusion into account ($A \gg 1$). This limiting case has been thoroughly discussed in the literature (see Ref. 15). When saturation is homogeneous, the increase in the absorbed power with the field intensity slows down as the depth $y(0)$ of the dip approaches unity. However, in strong fields the dip flattens out, and because of this, the increase in the absorbed power does not stop. The transition from the homogeneous saturation region to the dip-flattening regime occurs at $A \sim W^{1/4}$, when the increase in the absorbed power due to homogeneous saturation slows down so much that it becomes comparable to the increase due to dip flattening.

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APPENDIX

The correction to the argument of $\cot z$ in (23) related to the finite value of ε can become important in three cases:

1. ε is large (α is close to $1,2,\ldots,n,\ldots$).

2. The value of $\cot z$ is large (α is close to $5/2, 9/2, \ldots, 2n+1/2, \ldots$).

3. The value of $\cot z$ is small (α is close to $3/2, 7/2, \ldots, 2n-1/2, \ldots$).

Let $\alpha = n+\delta$, with $\delta \ll 1$. Then $\varepsilon \simeq (-1)^{n+1} C_n W^{2\alpha}/\delta^2$, with C_n positive (e.g., $C_1 = 3/64$), $\sin \pi\alpha \simeq (-1)^n \delta$, and $\cos \pi\alpha \simeq (-1)^n$. The argument of $\arctan z$ in (23) can be written as

$$\frac{\varepsilon \sin \pi\alpha}{1-\varepsilon \cos \pi\alpha} \simeq -\frac{\pi\delta C_n W^{2\alpha}}{C_n W^{2\alpha}+\delta^2}.$$

It is at its maximum at $\delta \sim W^{\alpha}$; the maximum value of the argument is of order $W^{\alpha} \ll 1$.

Let $\alpha = 2n+1/2+\delta$, with $\delta \ll 1$. Then $\varepsilon \simeq C_n W^{2\alpha}\delta$ (e.g., $C_1 = 2/675$), $\sin \pi\alpha \simeq 1$, and $\cos \pi\alpha \simeq 0$. The value of $\cot z$ in (23) is given by the approximate expression

$$\cot z \simeq \frac{1}{(\pi/2 + C_n W^{2\alpha})\delta},$$

i.e., the correction related to $\varepsilon \neq 0$ is small ($W \ll 1$). Note that if we put $n=0$ (a weak field), then $\varepsilon \simeq W^{2\alpha}$, i.e., ε is not small in terms of δ.

Finally, let $\alpha = 2n - 1/2 + \delta$, with $\delta \ll 1$. Then $\varepsilon \simeq -C_n W^{2\alpha}\delta$ (e.g., $C_1 = 2/9$), $\sin \pi\alpha \simeq -1$, and $\cos \pi\alpha \simeq 0$. The value of $\cot z$ in (23) is given by the approximate expression

$$\cot z \simeq -(\pi/2 + C_n W^{2\alpha})\delta,$$

i.e., the correction related to $\varepsilon \neq 0$ is small ($W \ll 1$).

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