Role of interaction in causing errors in optical soliton transmission

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We consider two solitons propagating under a filter-control scheme and describe the timing jitter that is caused by spontaneous-emission noise and enhanced by attraction between solitons. We find the bit-error rate as a function of system parameters (filtering and noise level), timing, initial distance, and the phase difference between solitons.

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In fiber-optic soliton systems, amplifier noise and interaction of solitons are the main sources of transmission errors. Both effects are very weak, and yet, in general, one cannot use a perturbation approach to obtain the error probability because errors occur when the signal changes substantially.[1–3] Order-unity changes in a nonlinear system usually are not described by Gaussian statistics. Moreover, it is not clear *a priori* when one may still consider the signal to be a soliton and when, on the contrary, fluctuations with a substantial change of shape contribute to the error probability. The method of obtaining the error probability that was suggested in Ref. 1 is a maximum-likelihood approach that boils down to finding an optimal fluctuation that brings a given large deviation of soliton parameters. The method is technically a saddle-point approximation in the path integral and is known to describe properly the tails of probability distributions that correspond to rare events.[4] A consistent development of the optimal fluctuation method for soliton-bearing systems was presented in Ref. 3, in which the conditions on the noise level and propagation distance were formulated for an optimal fluctuation to be close to a soliton with slowly changing parameters. This formulation made it possible to reduce the potentially infinite-dimensional problem to consideration of the finite set of soliton parameters and to effectively find the error probability for a single soliton propagating under different control schemes.[3] The probability-density function (PDF) is essentially Gaussian for Gordon–Haus jitter[5,6] and may have substantially non-Gaussian tails for different control schemes, notably the probability of signal loss with filtering and amplitude modulation.[3] In this Letter, we consider two interacting solitons. This is exactly the problem that was treated in a pioneering paper[1] in which phase fluctuations were neglected and a crude approximation of the optimal path was employed. Here we show that soliton interaction is so powerful in enhancing noise and so sensitive to the distance and the phase difference between solitons that one needs a quantitatively accurate description to find the error probability. We first describe analytically the PDF of the distance between solitons in the model used in Ref. 1. We show that the PDF is qualitatively similar but the error probability was grossly underestimated in Ref. 1. A true optimal fluctuation gives a flat PDF tail, and one needs to account for fluctuations around the optimal fluctuation to get a correct PDF form. Even more important is accounting for phase dynamics, which is shown below to decrease the error probability substantially. A not only quantitatively accurate but also physically transparent picture emerges from our analysis. As was already noted in Refs. 1 and 2, the PDF $\mathcal{P}(q)$ deviates from Gaussian at distances smaller than some value of $\bar{q}$. Here we show that crossover in the PDF happens at the point where interaction between solitons is comparable to noise, and we find the dependence of $\bar{q}$ on system parameters and the form of the non-Gaussian tail. A physically transparent picture emerges from our analysis: The role of noise is to bring solitons to distance $\bar{q}$, and then interaction takes over. Up to order-unity factor, $\mathcal{P}(q)$ at $q < \bar{q}$ is a Gaussian value taken at $\bar{q}$, multiplied by two factors. The first factor is simply the inverse relative speed of solitons at distance q (the faster the speed, the smaller the probability of finding them at a given distance). Under the filter-control scheme, the relative speed is proportional to the force between solitons, which is exponentially dependent on the distance. The second factor is the fraction of 2π occupied by the interval of phase differences that the solitons must have when they are separated by $\bar{q}$, which guarantees that attraction is not replaced with repulsion on the way from $\bar{q}$ to q. Again, because the interaction depends exponentially on the distance. This interval can be shown to be proportional to $\exp(q - \bar{q})$. The two factors thus contribute each e^q to the PDF, so the non-Gaussian tail is shown below to have an exponential form $\mathcal{P}(q) \propto e^{2q}$ at $q < \bar{q}$. We also numerically solve an exact Fokker–Planck equation for the PDF and find the error probability as a function of different parameters.

The distance q and phase difference ϕ between two solitons satisfy the following equations written in soliton units[7]:

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$$\ddot{q} + \gamma\dot{q} = -8e^{-q}\cos(\phi) + \xi\,, \quad (1)$$

$$\ddot{\phi} + \gamma\dot{\phi} = -8e^{-q}\sin(\phi) + \xi_\phi\,. \quad (2)$$

Here, γ gives the effect of the filter, the noise is white, $\langle\xi(0)\xi(t)\rangle = 2D\delta(t)$, $\langle\xi_\phi(0)\xi_\phi(t)\rangle = 2D_\phi\delta(t)$, with $D_\phi = D[3 + \gamma^2(1 + \pi^2/12)]/(1 + \pi^2\gamma^2/4)$.[8] We consider the propagation distance to be large compared with the distance between amplifiers and employ continuous description (a limit opposite that reported in Ref. 2). One can neglect the continuous spectrum of perturbations and consider only the variations of soliton parameters if noise is weak: $D \ll \sqrt{\gamma}$.[3] Following Ref. 1, we start from the worst case, with $\phi \equiv 0$, and initially neglect phase dynamics. Such a model, although it is shown below to be unrealistic, allows one to understand the basic physics involved. The Fokker–Planck equation for the joint PDF, $\mathcal{P}(q,\dot{q},t)$, follows from Eq. (1):

$$[\partial_t + \partial_q\dot{q} - \partial_{\dot{q}}(8e^{-q} + \gamma\dot{q}) - D\partial_{\dot{q}}{}^2]\mathcal{P}(q,\dot{q},t) = 0\,, \quad (3)$$

where the initial condition is $\mathcal{P}(q,\dot{q},0) = \delta(q - q_0)\delta(\dot{q})$.

At $\gamma t \gg 1$, inertia term $\ddot{q}$ can be neglected for all but rare realizations that correspond to the very fast approach of solitons. Neglecting $\ddot{q}$ gives the correct PDF everywhere except for a short (order-unity) interval of distances at which the approximation of exponential interaction breaks anyway. This can be seen in Fig. 1, in which curves 3 and 4 represent the solutions of Eqs. (3) and (4), respectively. Without the inertia term, Eq. (1) gives the Fokker–Planck equation for $\mathcal{P}(q,t)$:

$$(\gamma^2\partial_t - \gamma\partial_q 8e^{-q} - D\partial_q{}^2)\mathcal{P}(q,t) = 0\,, \quad (4)$$

which is exactly solvable in terms of the Whittaker function and gives, for $\mathcal{P}(q,0) = \delta(q - q_0)$,

$$\mathcal{P}(q,t) = \frac{\exp(\zeta - \zeta_0)/2}{\sqrt{\zeta\zeta_0}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\,k^2\cosh(\pi k)$$
$$\times\exp(-Dk^2t/\gamma^2)W_{-1/2,ik}(\zeta)W_{-1/2,-ik}(\zeta_0)\,,$$
$$\zeta = 8\gamma e^{-q}/D\,. \quad (5)$$

Equation (5) describes a Gaussian PDF, $\mathcal{P}(q) \propto \exp[-\gamma^2(q - q_0)^2/4Dt]$ at $q > \bar{q}(t) \equiv \ln(16t/\gamma q_0)$, and an exponential left-hand tail, $\mathcal{P}(q) \propto e^q$ at $q < \bar{q}(t)$. Noise and interaction are comparable at distance $\bar{q}(t)$, which can be estimated by substitution of the Gaussian distribution into Eq. (4) and comparing the second and the third terms. Note that $\bar{q}$ does not depend on the amplitude of the noise, whereas $\mathcal{P}(\bar{q})$, of course, depends on D. The form of the PDF at $q < \bar{q}$ is completely determined by the balance between filtering and interaction, which gives $\dot{q} = -8e^{-q}/\gamma$. The probability of finding a soliton in the interval $(q, q + \mathrm{d}q)$ is proportional to the time that is spends there, i.e., to $1/\dot{q} \propto e^q$. The shape of the PDF, $\mathcal{P}(q,t)$, at $q < \bar{q}$ does not depend on time, whereas its amplitude grows [as $\exp(-\gamma^2 q_0{}^2/4Dt)$].

We now consider the full system [Eqs. (1) and (2)], which also accounts for phase dynamics. Note that $\phi = 0$ is an unstable fixed point. Even when one starts from $\phi_0 = 0$, noise causes the phase to deviate from zero, and then interaction drives ϕ toward the stable point, $\phi = \pi$, which corresponds to repulsion. It is thus clear that the account of phase dynamics has to decrease the probability of solitons' approaching each other. We again neglect soliton inertia and obtain the following Fokker–Planck equation for the joint PDF, $\mathcal{P}(q,\phi,t)$:

$$[\gamma^2\partial_t + (\partial_\phi\sin\phi - \partial_q\cos\phi)8\gamma e^{-q}$$
$$- D\partial_q{}^2 - D_\phi\partial_\phi{}^2]\mathcal{P}(q,\phi,t) = 0\,. \quad (6)$$

Again, at $q < \bar{q}$, interaction dominates, and one may disregard the last two (noise) terms and find out that $\mathcal{P}(q,\phi,t) \propto e^{2q}$ turns the second (advective) term in Eq. (6) into zero. One may interpret this in the following way: To pass from $\bar{q}$ to some smaller q, one needs an attraction, that is, $\phi < \pi/2$ along the way. This requires that the phase at $q \simeq \bar{q}$ be exponentially small, since the trajectories in the ϕ–q plane are exponentially diverging near zero phase: $\phi \propto e^{-q}$. Therefore, only the exponentially small fraction of realizations corresponds to the motion from $\bar{q}$ to smaller q, which contributes the factor $\exp(q - \bar{q})$ to the probability. At $q > \bar{q}$, the PDF approaches Gaussian. To get the PDF everywhere, we solve Eq. (6) numerically with different ϕ_0.

In Fig. 1 we compare distance PDFs obtained by different methods. Curve 1 is Gaussian and is obtained without interaction. Curve 2 is taken from Ref. 1, that is, with phase fluctuations neglected and the optimal path crudely approximated. Curve 3 represents an accurate computation of the same model, that is, the numerical solution of Eq. (3). Curve 4 represents solution (5) of Eq. (4), which shows that inertia can indeed be neglected in this case for all practical purposes. Accounting for phase fluctuations, that is, solving Eq. (6), gives curves 5 and 6 for the initial phases $\phi_0 = 0$ and $\phi_0 = \pi$, respectively. Both curves follow $\mathcal{P}(q) \propto e^{2q}$ at $q \lesssim 7$. Note that in this region curve 5 indeed gives $\mathcal{P}$, which

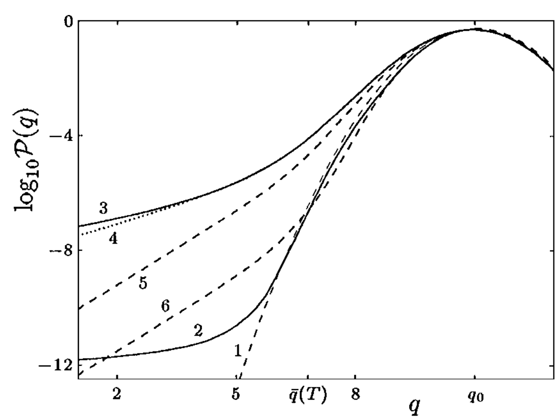

Fig. 1. Distance PDF $\mathcal{P}(q)$. The parameters are as in Ref. 1: $q_0 = 11$, $\gamma = 0.4$, $D = 0.0002$, and $T = 250$. See text for explanation of curves.

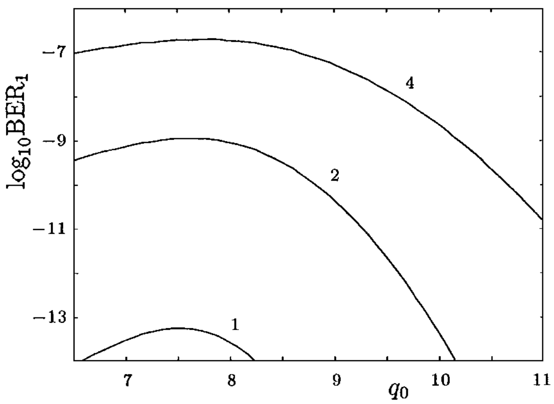

Fig. 2. BER_1 as a function of q_0. $T = 150$, $\gamma = 0.4$. See text for explanation of curves.

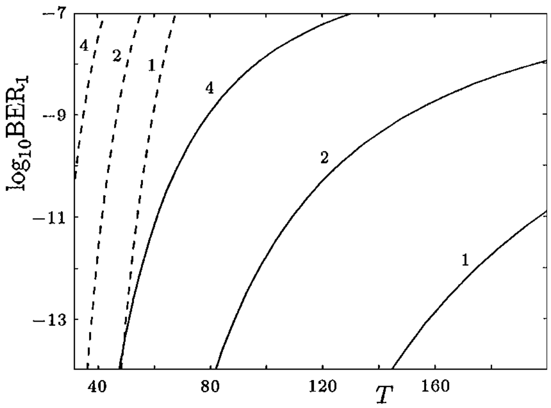

Fig. 3. BER_1 as a function of time. $q_0 = 8$, $\gamma = 0.4$. See text for explanation of curves.

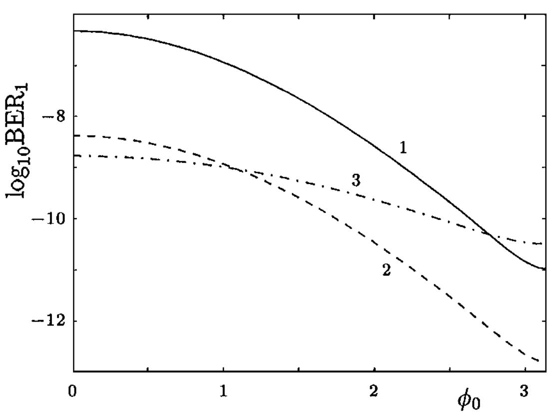

Fig. 4. BER_1 as a function of initial phase ϕ_0. $q_0 = 9.5$, $\gamma = 0.4$; curve 1, $D = 0.0003$, $T = 200$; curve 2, $d = 0.0006$, $T = 100$; curve 3, $d = 0.0048$, $T = 25$.

is $\exp(q - \bar{q})$ times smaller than that given by curve 4. One can see that phase fluctuations dramatically decrease the probability of solitons' approaching each other. Choosing initial phase $\phi_0 = \pi$ (corresponding to repulsion) reduces the probability even further.

Assuming that there is an error when the solitons approach closer than the threshold q_T, we calculate their contributions to the bit-error rate (BER_1): $E(q_T, q_0, \phi_0, T; \gamma, D) = \int_0^{q_T} \mathrm{d}q\, \mathcal{P}(q)$. The simplest to get is the dependence of E on the threshold, $E(q_T) \propto \exp(2q_T)$, as long as $q_T < \bar{q}$. For numerical evaluation, we take $q_T = 3$. The dependence of E on the other five parameters can be simplified by use of an exact rescaling symmetry, $e^q \to \lambda e^q$, $D \to D/\lambda$, $t \to \lambda t$, and two approximate symmetries, $e^q \to \lambda e^q$, $\gamma \to \gamma/\lambda$, $D \to D/\lambda^2$ and $\gamma \to \lambda\gamma$, $D \to \lambda D$, $t \to \lambda t$. One obtains these symmetries by neglecting the dependence $D_\phi(\gamma)$ (very weak at $\gamma < 1/2$). We are thus left with a function of three parameters, DT/γ^2, ϕ_0, and $q_0 - \bar{q}(T) = \ln(\gamma q_0 \exp(q_0)/16T)$. The dependences of E on T and q_0 are given mainly by the Gaussian probability of moving from q_0 to $\bar{q}$: $\ln E \sim \gamma^2(q_0 - \bar{q})^2/4DT$, which nicely fits the curves shown in Fig. 2 and [up to a slow logarithmic dependence, $\bar{q}(T)$], in Fig. 3. The dashed curves are for $\phi_0 = 0$, and the solid curves are for $\phi_0 = \pi$; $D = 0.0001$, $D = 0.0002$, and $D = 0.0004$ for the top, middle, and bottom curves, respectively.

Note that the curves contain a part where BER_1 increases with q_0. This is because at the beginning the solitons move away from each other because of their interaction, which takes some time. The effective time to reach $\phi = 0$ and small q is shorter, and the error probability decreases. The dependence $E(\phi_0)$, shown in Fig. 4, can be understood as follows: Solitons reach a distance less than $\bar{q}$ if their phase difference is near zero before reaching $\bar{q}$. Since ϕ changes from ϕ_0 to zero when the distance is larger than $\bar{q}$, the interaction is weak there, and the probability can be estimated as Gaussian: $E \propto \exp(-\phi_0{}^2\gamma^2/4D_\phi T)$. Figure 4 shows that the Gaussian factor satisfactorily describes the decrease of BER_1 with ϕ_0, except for a small area in the vicinity of $\phi_0 \approx \pi$.

An approximate estimate of the interaction-enhanced contribution of amplifier noise to the error probability is thus

$$\ln E(q_T, q_0, \phi_0, T; \gamma, D) \simeq 2(q_T - \bar{q}) - [q_0 - \bar{q}(T)]^2 \times \gamma^2/4DT - \phi_0{}^2\gamma^2/4D_\phi T\,. \tag{7}$$

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