Growth of Density Inhomogeneities in a Flow of Wave Turbulence

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We consider the flow being a superposition of random waves and describe the evolution of the spectrum of the passive scalar in the leading (fourth) order with respect to the wave amplitudes. We find that wave turbulence can produce an exponential growth of the passive scalar fluctuations when either both solenoidal and potential components are present in the flow or there are potential waves with the same frequencies but different wave numbers.

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The action of a random flow on a passive substance it carries involves a diverse set of phenomena depending on the scale and subject under consideration [1]. Every single fluid particle undergoes diffusion on a time scale exceeding the velocity correlation time. The distance between two particles generally grows exponentially (with the rate called Lyapunov exponent) at the scales smaller than the correlation scale of velocity gradients [1]. In a random compressible flow, the asymptotic in time rate of an infinitesimal volume change along the trajectory is given by the sum of Lyapunov exponents which is generally nonpositive for the simple reason that contracting regions contain more fluid particles and thus have more statistical weight [2,3]. After averaging over the set of trajectories, the sum of the Lyapunov exponents gives the asymptotic rate of entropy production. If the sum is strictly negative, then inhomogeneities in the passive density grow exponentially and it tends to concentrate on a fractal set (so-called Sinai-Ruelle-Bowen measure) [4,5]. Those phenomena have been studied mainly for random flows delta correlated in time and for dynamical systems (see [1,6,7] and the references therein).

In this Letter, we consider the flow that is a superposition of waves of small amplitude (see, e.g., [8–15]). In the first order with respect to the wave amplitude, the motion of every fluid particle is a superposition of purely periodic oscillations. In the second order, every wave provides for the Stokes drift of fluid particles along the wave vector. The relative motion of fluid particles appears because of interference of drifts produced by different waves. The statistics of the distance between two particles can be described in terms of the pair correlation function of the passive scalar (see, e.g., [1]). Here we derive the equation for the pair correlation function of passive density up to the fourth order with respect to the wave amplitudes. At the scales much larger than the wavelengths this equation describes diffusion. Considering the scales smaller than the wavelengths, we show that in this order nonzero Lyapunov exponents appear in two and three dimensions both for potential and solenoidal waves. We also find the conditions for a nonzero sum of the Lyapunov exponents which provides for an exponential growth of density inhomogeneities: Either the medium allows for waves with the same frequency but different wave numbers or waves must have both solenoidal and potential component. That can be interpreted as follows: The leading-order contribution comes from the pair of waves having coinciding frequencies but producing different Stokes drifts.

Consider the continuity equation for the passive scalar density $\phi(\mathbf{r}, t)$ (e.g., pollutant)

$$\dot{\phi} + \mathrm{div}(\mathbf{v}\phi) = 0. \quad (1)$$

Let the fluid velocity field be a superposition of waves with the dispersion law $\omega = \Omega_{\mathbf{k}}$:

$$\mathbf{v}(\mathbf{r}, t) = \int \mathbf{A}_{\mathbf{k},\omega} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\, d\mathbf{k}\, d\omega; \quad (2)$$

$$\sqrt{2}\mathbf{A}_{\mathbf{k},\omega} = \mathbf{c}_{\mathbf{k}}\delta(\omega - \Omega_{\mathbf{k}}) + \mathbf{c}^*_{-\mathbf{k}}\delta(\omega + \Omega_{-\mathbf{k}}). \quad (3)$$

When wave amplitudes are small, wave turbulence is expected to have statistics close to Gaussian [16]. The respective small parameter $\epsilon = kc_k/\Omega_k \ll 1$ is the ratio of the fluid velocity to the wave velocity, or the ratio of the oscillation amplitude of fluid particles to the wavelength. That parameter must be small for a wave to exist; wave breaking generally occurs for ϵ well below unity. The wave amplitudes are thus taken as random Gaussian variables with zero mean and covariance $\langle A^\alpha_{\mathbf{k},\omega} A^\beta_{\mathbf{k}',\omega'}\rangle = E^{\alpha\beta}_{\mathbf{k},\omega}\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')$, where $2E^{\alpha\beta}_{\mathbf{k},\omega} = \varepsilon^{\alpha\beta}_{\mathbf{k}}\delta(\omega - \Omega_{\mathbf{k}}) + \varepsilon^{\alpha\beta}_{-\mathbf{k}}\delta(\omega + \Omega_{-\mathbf{k}})$. Here $\alpha, \beta = 1, \ldots, d$. Physically, we assume all averages to be done over space with the dimensionality $d = 2, 3$. Note that we neglect finite frequency width, assuming the attenuation rate to be smaller than $\Omega\epsilon$. Possible non-Gaussianity of wave statistics depends on the types of waves and will be considered elsewhere. The theory presented here is general; the waves can be sound waves, gravity waves (surface or internal), inertial-gravity waves, Rossby waves, etc.

In the Fourier representation, $\phi(\mathbf{r}, t) = \int f_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k}$ ($f_{\mathbf{k}} = f^*_{-\mathbf{k}}$), the continuity Eq. (1) has the form $\dot{f}_1 = -i\int \mathbf{k}_1^\alpha e^{-i\omega_a t} f_2 A_a^\alpha \delta_{-12a} d_{2a}$. Here and below, instead of wave vectors and frequencies we keep their labels; e.g., $f_2 = f_{\mathbf{k}_2}$ $A_a = A_{\mathbf{k}_a,\omega_a}$, $\delta_{-12a} = \delta(-\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_a)$, and $d_{2a} = d\mathbf{k}_2 d\mathbf{k}_a d\omega_a$. Integration over number indices includes integration over wave vectors while that over letter indices $a, b, \ldots$ also includes integration over the frequencies. Repeated Greek indices imply summation.

We shall now derive the equation for the pair correlation function perturbatively in ϵ. That can be done in a straightforward perturbation theory solving the equations of motion for the hierarchy of correlation functions. We apply a much more efficient and compact formalism of statistical near-identity transformations [12,13]. Consider first the equation for the quantity $F_{12}(t) = f_1(t) f_2^*(t)$:

$$\dot{F}_{12} = \int U^\alpha_{1234a} e^{-i\omega_a t} F_{34} A_a^\alpha\, d_{34a},$$
$$U^\alpha_{1234a} = -ik_1^\alpha \delta_{-13a}\delta_{-24} - ik_2^\alpha \delta_{-24-a}\delta_{-13}. \tag{4}$$

The transformation to the new variable H,

$$F_{12} = H_{12} + \int U^\alpha_{1234a} \frac{e^{-i\omega_a t}}{-i\omega_a} H_{34} A_a^\alpha d_{34a}, \tag{5}$$

eliminates the term linear in A

$$\frac{\partial H_{12}}{\partial t} = \frac{1}{2}\int W^{\alpha\beta}_{1256ab} e^{-i(\omega_a+\omega_b)t} H_{56} A_a^\alpha A_b^\beta d_{56ab},$$
$$W^{\alpha\beta}_{1256ab} = \frac{1}{-i\omega_b}\int U^\alpha_{1234a} U^\beta_{3456b} d_{34} + \{(a,\alpha) \leftrightarrow (b,\beta)\}. \tag{6}$$

We assume that $E_{\mathbf{k},\omega} \to 0$ as $\omega \to 0$, so that (5) contains no small denominator. We neglected the integral with time derivative $\dot{H}$, which is of higher order in ϵ (more precisely, it will have no small denominator after the second transformation so it is nonresonant). What we have done is equivalent to the first order of perturbation theory in ϵ^2. Averaging (6), one gets zero so that we ought to go to the next order. This can be done by introducing yet another new variable G:

$$H_{12} = G_{12} + \int W^{\alpha\beta}_{1256ab} \frac{e^{-i(\omega_a+\omega_b)t} - 1}{-2i(\omega_a + \omega_b)} \times G_{56}[A_a^\alpha A_b^\beta - E_a^{\alpha\beta}\delta_{ab}] d_{56ab}, \tag{7}$$

where $\delta_{ab} = \delta(\mathbf{k}_a + \mathbf{k}_b)\delta(\omega_a + \omega_b)$. Since this transformation, unlike (5), has a small denominator, then we have chosen the time antiderivative on the right-hand side (rhs) of (7) so that the numerator also vanishes when $\omega_a + \omega_b = 0$. The transformation (7) "pushes" the randomness (the fluctuating part that averages to zero) to the order ϵ^6. Neglecting these terms, we find the dynamic equation [17]

$$\dot{G}_{12} = \frac{1}{2}\int W_{1234a-a} G_{34} E_a d_{34a} + \frac{1}{2}\int W^{\alpha\beta}_{1234ab} \times W^{\mu\nu}_{3456-a-b} \frac{1 - \exp\{-i(\omega_a + \omega_b)t\}}{i(\omega_a + \omega_b)} \times G_{56} E_a^{\alpha\mu} E_b^{\beta\nu} d_{3456ab}, \tag{8}$$

where index $-a$ stands for $(-\mathbf{k}_a, -\omega_a)$. This equation is now ready to be averaged over the statistics of waves. Statistical space homogeneity is enforced by the delta functions, $\langle G_{12}\rangle = N_1\delta(\mathbf{k}_1 - \mathbf{k}_2)$, and we obtain

$$\dot{N}_1 = \pi\int (k_1^\alpha k_a^\beta - k_1^\beta k_b^\alpha)(k_1^\mu k_a^\nu - k_1^\nu k_b^\mu) \times E_a^{\alpha\mu} E_b^{\beta\nu}(N_5 - N_1)\frac{1}{\omega_a^2}\delta(\omega_a + \omega_b)\delta_{-15ab} d_{5ab}. \tag{9}$$

Since N is real, and W is purely imaginary, the first integral in Eq. (8), when $\mathbf{k}_1 = \mathbf{k}_2$, should vanish, and in the second integral only the real part survives. Note that the real part of the quotient in (8) is $(\omega_a + \omega_b)^{-1}\sin(\omega_a + \omega_b)t \to \pi\delta(\omega_a + \omega_b)$ as $t \to \infty$ which gives the frequency delta function in (9) at $t \gg \omega_a, \omega_b$. On the resonance manifold $\omega_a + \omega_b = 0$, the expression for the kernel W is simplified: $i\omega_a W^{\alpha\beta}_{1256ab} = (k_1^\alpha k_a^\beta - k_1^\beta k_b^\alpha)\delta_{-15ab}\delta_{-26} - (k_2^\alpha k_a^\beta - k_2^\beta k_b^\alpha)\delta_{-26-a-b}\delta_{-15}$. Finally, to arrive at Eq. (9), one uses the symmetry $E_a = E_{-a}$.

The pair correlation function of the original passive density ϕ is a Fourier transform of $N_{\mathbf{k}}$ up to the terms of higher order in ϵ: $\langle\phi(\mathbf{r}_1, t)\phi(\mathbf{r}_2, t)\rangle = \int N_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}\, d\mathbf{k}$.

For the even and positive dispersion law ($\Omega_{\mathbf{k}} = \Omega_{-\mathbf{k}} > 0$), Eq. (9) becomes

$$\dot{N}_1 = \frac{\pi}{2}\int (k_1^\alpha k_a^\beta + k_1^\beta k_b^\alpha)(k_1^\mu k_a^\nu + k_1^\nu k_b^\mu) \times \varepsilon_a^{\alpha\mu}\varepsilon_b^{\beta\nu}(N_5 - N_1)\frac{1}{\Omega_a^2}\delta(\Omega_a - \Omega_b) \times \delta_{-15a-b}\, d\mathbf{k}_5\, d\mathbf{k}_a\, d\mathbf{k}_b. \tag{10}$$

In 1D, the rhs of this equation vanishes because, due to the frequency delta function, either $\mathbf{k}_a = \mathbf{k}_b$ (and then $\mathbf{k}_5 = \mathbf{k}_1$), or $\mathbf{k}_a = -\mathbf{k}_b$ (and then the kernel vanishes).

Let us describe some general properties of Eq. (10) which has a form of the kinetic equation for elastic scattering. One can readily show that the kernel in (9) is non-negative: For any point $(\mathbf{k}_a, \omega_a, \mathbf{k}_b, \omega_b)$ in the coordinates that makes $E_a^{\alpha\mu}$ and $E_b^{\beta\nu}$ diagonal, one has

$$(k_1^\alpha k_a^\beta - k_1^\beta k_b^\alpha)(k_1^\mu k_a^\nu - k_1^\nu k_b^\mu)E_a^{\alpha\mu}E_b^{\beta\nu} = \sum_{\alpha,\beta}(k_1^\alpha k_a^\beta - k_1^\beta k_b^\alpha)^2 E_a^{\alpha\alpha}E_b^{\beta\beta} \geq 0. \tag{11}$$

Because of this fact, Eq. (9) satisfies the maximum principle (such as the diffusion equation): If for some numbers m and M at some instant t_0 , $m \leq N_{\mathbf{k}}(t_0) \leq M$ for all $\mathbf{k}$, then also for any $t > t_0$, $m \leq N_{\mathbf{k}}(t) \leq M$ for all $\mathbf{k}$.

Consider now $k_1 \ll k_5$ in (10) that is harmonics of passive scalar with the wave numbers smaller than all the wave numbers of wave turbulence. Assuming that $N_5 \ll N_1$, we get the decay of small-k harmonics: $\dot{N}_1 = -N_1 k_1^\alpha k_1^\beta D_{\alpha\beta}$. By virtue of (11), $D > 0$, so that at the scales much larger than the wavelengths our equation describes usual diffusion discussed previously [8–15].

Diffusion and decay of large-scale harmonics of the density is only one side of the story. As we show now, small-scale fluctuations may grow in such a flow. Let us integrate (9) over $\mathbf{k}_1$, split the integral into a difference of two integrals with N_5 and with N_1, change the integration variables in the integral with N_5 ($1 \leftrightarrow 5, a \to -a, b \to -b$), and notice that the terms linear in $\mathbf{k}_1$ disappear. In so doing, we find a closed equation for the mean square density of the passive scalar, $\mathcal{N}(t) = \langle \phi(\mathbf{r}, t)^2 \rangle = \int N_\mathbf{k}\, d\mathbf{k}$:

$$\begin{aligned} \dot{\mathcal{N}} &= \lambda \mathcal{N}, \\ \lambda &= \pi \int (k_a^\alpha k_a^\beta - k_b^\alpha k_b^\beta)(k_a^\mu k_a^\nu - k_b^\mu k_b^\nu) \\ &\quad \times E_a^{\alpha\mu} E_b^{\beta\nu} \omega_a^{-2} \delta(\omega_a + \omega_b) d_{ab}. \end{aligned} \tag{12}$$

The growth rate λ is non-negative by virtue of (11). Growth of $\langle \phi^2 \rangle$ (with $\langle \phi \rangle$ fixed) means that the density is concentrated in smaller and smaller regions leaving depletions of growing volume.

If the flow is solenoidal then the single-point moments of the density do not change and, in particular, $\lambda \equiv 0$. If the flow is purely potential, $E_{\mathbf{k},\omega}^{\alpha\beta} = P_{\mathbf{k},\omega} k^\alpha k^\beta k^{-2}$, then

$$\lambda = \pi \int (k_a^2 - k_b^2)^2 \frac{(\mathbf{k}_a \cdot \mathbf{k}_b)^2}{k_a^2 k_b^2 \omega_a^2} \delta(\omega_a + \omega_b) P_a P_b d_{ab}. \tag{13}$$

Since $P_{\mathbf{k},\omega} \propto \delta(\omega \pm \Omega_{\mp\mathbf{k}})$, then (13) contains $(k_a^2 - k_b^2)^2 \delta(\Omega_a \pm \Omega_b)$. We thus conclude that, for purely potential waves, squared density growth rate appears in the ϵ^4 order only if there are waves with the same frequencies but different wave numbers.

Now let us consider the most common case $\Omega_\mathbf{k} = \Omega_k > 0$. As seen from (13), for purely potential waves with an isotropic dispersion law (such as sound or gravity-capillary surface waves), the growth rate is zero in the ϵ^4 order. We now assume the turbulence spectrum to be a sum of potential and solenoidal isotropic components:

$$\varepsilon_\mathbf{k}^{\alpha\beta} = p_k \frac{k^\alpha k^\beta}{k^2} + s_k \left(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right). \tag{14}$$

Then the growth rate (12) is proportional to the product of the solenoidal and potential components:

$$\lambda = \Delta_d \int_0^\infty p(k) s(k) \frac{k^{2+2d}}{\Omega_k^2} \left| \frac{d\Omega}{dk} \right|^{-1} dk, \tag{15}$$

where $\Delta_2 = 2\pi^3$ in 2D, and $\Delta_3 = 16\pi^3/3$ in 3D.

The growth of the integral $\int N_\mathbf{k}\, d\mathbf{k}$ despite the decay of small-k harmonics means that large-k harmonics grow. Let us consider now the evolution of the density spectrum at the scales much smaller than the wavelengths: $k_a, k_b \ll k_1, k_5$. In this limit, we can replace $N_{\mathbf{k}-\mathbf{k}_a+\mathbf{k}_b} - N_\mathbf{k}$ by

$$\frac{\partial N_k}{\partial k} \frac{[\mathbf{k} \times (\mathbf{k}_a - \mathbf{k}_b)]^2}{k^3 |\mathbf{k}_a - \mathbf{k}_b|^2} + \frac{\partial^2 N_k}{\partial k^2} \frac{[\mathbf{k} \cdot (\mathbf{k}_a - \mathbf{k}_b)]^2}{k^2 |\mathbf{k}_a - \mathbf{k}_b|^2} \tag{16}$$

in (10). Higher-order derivatives give small contributions so that (10) turns into the differential equation of the second order. In an isotropic case it has a general form

$$\frac{\partial N_k}{\partial t} = Ak \frac{\partial N_k}{\partial k} + Bk^2 \frac{\partial^2 N_k}{\partial k^2}. \tag{17}$$

Particularly, in two dimensions we derive the following from (9), (14), and (16):

$$A = 3\tilde{\omega}_{pp} + 15\tilde{\omega}_{ss} + 14\tilde{\omega}_{ps}, \qquad B = \tilde{\omega}_{pp} + 5\tilde{\omega}_{ss} + 10\tilde{\omega}_{ps}, \qquad \tilde{\omega}_{pp} = \int_0^\infty p_k^2 \mu_k\, dk,$$

$$\tilde{\omega}_{ss} = \int_0^\infty s_k^2 \mu_k\, dk, \qquad \tilde{\omega}_{ps} = \int_0^\infty p_k s_k \mu_k\, dk, \qquad \mu_k = \frac{\pi^3}{16} \frac{k^6}{\Omega_k^2} \left| \frac{d\Omega}{dk} \right|^{-1}.$$

In terms of variables $\tau = Bt$ and $x = \ln k$, (17) is a second order partial differential equation with constant coefficients

$$\frac{\partial N(x, \tau)}{\partial \tau} = (a - 1) \frac{\partial N(x, \tau)}{\partial x} + \frac{\partial^2 N(x, t)}{\partial x^2}, \tag{18}$$

which turns into the diffusion equation in a moving reference frame: $N_\tau = N_{\xi\xi}$ for $\xi = x + (a - 1)\tau$. Here $a = A/B$. We, thus, see that small-scale harmonics of the passive density undergo diffusion in k space (in logarithmic coordinates) in contradistinction to large-scale harmonics that diffuse in r space. It is this diffusion in k space which is responsible for the growth of density inhomogeneities. The coefficient a is determined by the relation between potential and solenoidal parts of the flow, and one can show that $1 < a \leq 3$. Since $a > 1$, the distribution always shifts to small k. If the flow is purely potential or purely solenoidal, then $a = 3$, and the differential approximation (17) conserves the quantity $\mathcal{N} = 2\pi \int N_k k\, dk$, similar to the original integral Eq. (10). The differential approximation corresponds to the so-called Batchelor regime of passive scalar (see, e.g., [1]). Comparing (17) with the results of [1,18], one can find the Lyapunov exponents:

$$\lambda_1 \propto \tilde{\omega}_{pp} + 5\tilde{\omega}_{ss} - 6\tilde{\omega}_{ps}, \quad \lambda_2 \propto -\tilde{\omega}_{pp} - 5\tilde{\omega}_{ss} - 26\tilde{\omega}_{ps}.$$

If $\omega_{ps} \neq 0$, the sum $\lambda_1 + \lambda_2$ is nonzero which signals the

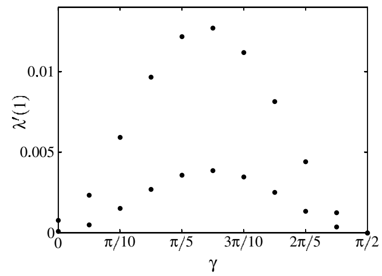

FIG. 1. Logarithmic growth rate as a function of the polarization angle for two levels of the wave energy.

development of an intermittent density field. One can characterize the statistics of such a field, for instance, considering density moments: $\langle\phi^\alpha\rangle \propto \exp[\lambda(\alpha)t]$. The function $\lambda(\alpha)$ is convex and has zeros at $\alpha = 0, 1$ [1,3]. The (negative) derivative at zero is the decay rate of the mean of $\log\phi$ equal to the sum of the Lyapunov exponents. The derivative at unity is the growth rate of the Lagrangian mean of $\log\phi$. For the Kraichnan model (of short-correlated velocity), the function is parabolic: $\lambda(\alpha) \propto \alpha(\alpha - 1)$ [1,19].

We have studied the growth of the Lagrangian mean of $\log\phi$ numerically. Both velocity and passive scalar were on the 2D torus of size 2π. The energy spectrum $\varepsilon_{\mathbf{k}}^{\alpha\beta}$ was nonzero inside the ring $4 < |\mathbf{k}| < 16$ with the dispersion law $\Omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|}$. We start with $\phi(\mathbf{r}, t = 0) \equiv 1$. After random choice of $\mathbf{c}_{\mathbf{k}}$ according to its Gaussian statistics and of initial position $\mathbf{R}(t = 0)$ (uniformly on torus), we compute the Lagrangian trajectory $\mathbf{R}(t)$ and the value of passive scalar $\Phi(t) = \phi[\mathbf{R}(t), t]$ on it. In a typical realization, $\Phi(t)$ grows exponentially in time. The logarithmic growth rate (averaged over realizations) is $\log[\Phi(t)]/t = \lambda'(1)$ and it is shown in Fig. 1 as a function of γ which is the angle between polarization vector $\mathbf{c}_{\mathbf{k}}$ and wave vector $\mathbf{k}$. Two groups of points (upper and lower) correspond to the same shape of energy spectrum $\varepsilon_{\mathbf{k}}^{\alpha\beta}$; the energy for upper points is twice larger. For purely potential waves ($\gamma = 0$), we observe a nonzero growth rate for higher amplitudes which means that it must appear in the next orders in wave amplitudes.

In conclusion, we have presented an analytic theory for the evolution of the second moment of the passive density in the lowest nonvanishing order (fourth) in wave amplitudes. We described the decay of large-scale and growth of small-scale harmonics of the density. Most important, we derived the conditions on the wave turbulence that provide for the growth of scalar fluctuations. Our numerics support the conclusions of the analytic theory. The description of the fluctuation growth in wave turbulence is an important step in the general theory of intermittency far from equilibrium and presents an important tool for (mainly geophysical) applications. The phenomena we described can be quantitatively characterized by the three numbers: the eddy diffusivity due to wave turbulence D, the largest Lyapunov exponent λ_1, and the growth rate of the squared density λ. For applications, let us give rough estimates for not very wide wave turbulence spectrum with typical v, q, Ω_q: $D \sim v^4 q^2 \Omega_q^{-3}$, $\lambda_1 \sim v^4 q^4 \Omega_q^{-3}$, and $\lambda \sim \langle(\mathrm{div}\,\mathbf{v})^2\rangle\langle(\mathrm{curl}\,v)^2\rangle\Omega_q^{-3}$.

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