Example of solving RC circuit by Laplace transform

Consider a series RC circuit with a known time dependent input voltage $V(t)$. We will determine the charge Q on the capacitor as a function of time. The charge is governed by the following differential equation.

$$V(t) = R\frac{dQ}{dt} + \frac{1}{C}Q$$

Taking a Laplace transform of both sides we get

$$\begin{aligned} L\{V\}(s) =& RL\left\{\frac{dQ}{dt}\right\}(s) + \frac{1}{C}L\{Q\}(s) \\ =& R\left(-Q(0) + sL\{Q\}(s)\right) + \frac{1}{C}L\{Q\}(s) \\ =& -RQ(0) + (Rs + 1/C)L\{Q\}(s) \\ \implies L\{Q\}(s) =& \frac{L\{V\}(s) + RQ(0)}{Rs + 1/C} \end{aligned}$$

We can now recover the charge as a function of time by inverting the Laplace transform. Here is a table of the inverse Laplace transforms we will use in this example:

$Y(s)$	$L^{-1}\{Y\}(t)$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s}e^{-as}$	$u_a(t)$
$e^{-as}Y(s)$	$u_a(t)L^{-1}\{Y\}(t-a)$

Case 1: Input voltage is zero

In this case,

$$\begin{aligned} L\{Q\}(s) =& \frac{RQ(0)}{Rs + 1/C} \\ =& \frac{Q(0)}{s - (-1/RC)}. \end{aligned}$$

Now take an inverse Laplace transform. We use the first row of our inverse transform table with $a = -1/RC$.

$$\begin{aligned} \implies Q(t) =& Q(0)L^{-1}\left\{\frac{1}{s - (-1/RC)}\right\} \\ =& Q(0)e^{-t/RC}. \end{aligned}$$

Case 2: Input voltage is unit step function

Now let's suppose that the input voltage is a unit step function at some time τ. The Laplace transform of this step function is

$$L\{u_\tau\}(s) = \frac{1}{s}e^{-\tau s},$$

So our equation for the Laplace transform of Q becomes

$$\begin{aligned} L\{Q\}(s) =& \frac{e^{-\tau s}/s + RQ(0)}{Rs + 1/C} \\ =& e^{-\tau s}\frac{1}{s(Rs+1/C)} + \frac{Q(0)}{s-(-1/RC)} \end{aligned}$$

Notice that we already took the inverse transform of the second term when we solved the problem with $V(t) = 0$. Take an inverse Laplace transform; use the $V(t) = 0$ solution; then apply the third rule in our table:

$$\begin{aligned} Q(t) =& L^{-1}\left\{e^{-\tau s}\frac{1}{s(Rs+1/C)}\right\} + L^{-1}\left\{\frac{Q(0)}{s-(-1/RC)}\right\} \\ =& L^{-1}\left\{e^{-\tau s}\frac{1}{s(Rs+1/C)}\right\} + Q(0)e^{(-1/RC)t} \\ =& u_\tau(t)L^{-1}\left\{\frac{1}{s(Rs+1/C)}\right\}(t-\tau) + Q(0)e^{(-1/RC)t} \end{aligned}$$

Almost done– do a partial fraction decomposition, then finish inverse transform using first rule in table.

$$\begin{aligned} \frac{1}{s(Rs+1/C)} =& \frac{C}{s} - \frac{C}{s-(-1/RC)} \\ \implies L^{-1}\left\{\frac{1}{s(Rs+1/C)}\right\} =& CL^{-1}\left\{\frac{1}{s}\right\} - CL^{-1}\left\{\frac{1}{s-(-1/RC)}\right\} \\ =& Ce^0 - Ce^{(-1/RC)t} \\ =& C - Ce^{-t/RC} \end{aligned}$$

Done. We have calculated the following formula for the charge as a function of time:

$$\begin{aligned} Q(t) =& u_\tau(t)L^{-1}\left\{\frac{1}{s(Rs+1/C)}\right\}(t-\tau) + Q(0)e^{(-1/RC)t} \\ =& u_\tau(t)\left(C - Ce^{-(t-\tau)/RC}\right) + Q(0)e^{-t/RC} \end{aligned}$$

Example of solving underdamped LRC circuit by Laplace transform

Now let's add an inductor, so that we have a series LRC circuit. Since we've been using L for the Laplace transform operator, we will denote the inductance of our circuit with a lowercase l. The voltage equation now reads

$$V(t) = l\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q$$

Taking a Laplace transform, we have

$$\begin{aligned} L\{V\}(s) =& l\left(-Q'(0) - sQ(0) + s^2L\{Q\}(s)\right) + R\left(-Q(0) + sL\{Q\}(s)\right) + \frac{1}{C}L\{Q\}(s) \\ =& -(ls+R)Q(0) - lQ'(0) + (ls^2 + Rs + 1/C)L\{Q\}(s) \end{aligned}$$

It's easy to solve for the Laplace transform of Q.

$$L\{Q\}(s) = \frac{L\{V\}(s) + (s+R/l)Q(0) + Q'(0)}{(s^2 + Rs/l + 1/lC)}$$

Case 1: Input voltage is zero

We'll need a few more entries in our Laplace transform table:

$Y(s)$	$L^{-1}\{Y\}(t)$
$\frac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos(bt)$
$\frac{b}{(s-a)^2+b^2}$	$e^{at}\sin(bt)$

We can "complete the square" to make express our function in terms of quantities appearing in our inverse Laplace transform table. Let's assume that $\frac{1}{lC} > \left(\frac{R}{2l}\right)^2$ so that we can take a real number square root in the following calculation. In this case, the circuit is said to be underdamped; this condition is sometimes expressed as $\frac{R}{2}\sqrt{\frac{C}{l}} < 1$ (these conditions are equivalent).

$$\begin{aligned}
L\{Q\}(s) =& \frac{(s+R/l)Q(0)+Q'(0)}{(s^2+Rs/l+1/lC)} \\
=& \frac{(s+R/l)Q(0)+Q'(0)}{(s-R/2l)^2-(R/2l)^2+1/lC} \\
=& Q(0)\frac{(s+\frac{R}{2l})}{\left(s-\frac{R}{2l}\right)^2+\left(\sqrt{\frac{1}{lC}-\left(\frac{R}{2l}\right)^2}\right)^2} + \frac{Q'(0)+\frac{R}{2l}Q(0)}{\sqrt{\frac{1}{lC}-\left(\frac{R}{2l}\right)^2}}\frac{\sqrt{\frac{1}{lC}-\left(\frac{R}{2l}\right)^2}}{\left(s-\frac{R}{2l}\right)^2+\left(\sqrt{\frac{1}{lC}-\left(\frac{R}{2l}\right)^2}\right)^2}.
\end{aligned}$$

Now take an inverse Laplace transorm.

$$\begin{aligned}
Q(t) =& Q(0)L^{-1}\left\{\frac{s-A}{(s-A)^2+B^2}\right\} + \frac{Q'(0)+\frac{R}{2l}Q(0)}{\sqrt{\frac{1}{lC}-\left(\frac{R}{2l}\right)^2}}L^{-1}\left\{\frac{B}{(s-A)^2+B^2}\right\} \\
=& Q(0)e^{At}\cos(Bt) + \frac{Q'(0)+\frac{R}{2l}Q(0)}{\sqrt{\frac{1}{lC}-\left(\frac{R}{2l}\right)^2}}e^{At}\sin(Bt)
\end{aligned}$$

Finally, we have calculated $Q(t)$.

$$Q(t) = Q(0)e^{-(R/2l)t}\cos\left(t\sqrt{\frac{1}{lC}-\left(\frac{R}{2l}\right)^2}\right) + \frac{Q'(0)+\frac{R}{2l}Q(0)}{\sqrt{\frac{1}{lC}-\left(\frac{R}{2l}\right)^2}}e^{-(R/2l)t}\sin\left(t\sqrt{\frac{1}{lC}-\left(\frac{R}{2l}\right)^2}\right)$$

Case 1: Arbitrary input voltage

Recall the convolution formula for the inverse Laplace transform of a product. If $L\{f\} = F$ and $L\{g\} = G$, then

$$L^{-1}\{F(s)G(s)\} = [u_0(t)f(t)] * [u_0(t)g(t)]$$

The square parenthesis are just indicating order of operations. We solve using an inverse Laplace transform.

$$\begin{aligned}
L\{Q\}(s) =& \frac{L\{V\}(s)+(s+R/l)Q(0)+Q'(0)}{s^2+Rs/l+1/lC} \\
\implies Q(t) =& L^{-1}\left\{L\{V\}(s)\frac{1}{s^2+Rs/l+1/lC}\right\} + L^{-1}\left\{\frac{(s+R/l)Q(0)+Q'(0)}{s^2+Rs/l+1/lC}\right\} \\
=& [u_0(t)V(t)] * \left[u_0(t)L^{-1}\left\{\frac{1}{s^2+Rs/l+1/lC}\right\}\right] + L^{-1}\left\{\frac{(s+R/l)Q(0)+Q'(0)}{s^2+Rs/l+1/lC}\right\}
\end{aligned}$$

We already calculated the second inverse transform in this formula. The first one is done similarly; we get

$$Q(t) = [u_0(t)V(t)] * \left[\frac{u_0(t)}{\sqrt{\frac{1}{lC} - \left(\frac{R}{2l}\right)^2}} \sin\left(t\sqrt{\frac{1}{lC} - \left(\frac{R}{2l}\right)^2} \right) \right]$$

$$+ Q(0)e^{-(R/2l)t} \cos\left(t\sqrt{\frac{1}{lC} - \left(\frac{R}{2l}\right)^2} \right) + \frac{Q'(0) + \frac{R}{2l}Q(0)}{\sqrt{\frac{1}{lC} - \left(\frac{R}{2l}\right)^2}} e^{-(R/2l)} \sin\left(t\sqrt{\frac{1}{lC} - \left(\frac{R}{2l}\right)^2} \right)$$