Final Exam Review problems solutions
Math 464 - Fall 2018

1. An urn has 1 red ball and 2 green balls. I draw a ball. If if it is red, I put it back in the urn. If it is green I do not put it back. Then I draw a second ball.

(a) Find the probability the second ball is red.

Solution: It helps to draw a tree.

$$\begin{aligned} P(2=R) &= P(2=R|1=R)P(1=R) + P(2=R|1=G)P(1=G) \\ &= \frac{1}{3} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{3} = \frac{4}{9} \end{aligned}$$

(b) Someone does this experiment and tells us he got a red ball on the second draw. What is the probability he got a red ball on the first draw?

Solution:

$$P(1=R|2=R) = \frac{P(1=R, 2=R)}{P(2=R)} = \frac{1/9}{4/9} = \frac{1}{4}$$

(c) Are the events "first ball is red" and "second ball is red" independent? Explain your reasoning. **Solution:** $P(1=R) = 1/3$. This is not equal to $P(1=R|2=R)$, so they are not independent.

2. Let X have the gamma distribution with $\lambda = 1/2$ and $w = 1/2$. Find the probability density function (pdf) of $Y = \sqrt{X}$. **Solution:**

$$F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y) = P(X \le y^2) = \int_0^{y^2} \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma(\frac{1}{2})} x^{-1/2} e^{-x/2}$$

Differentiate with respect to y to get the pdf of Y and use the second fundamental theorem of calc:

$$f_Y(y) = \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma(\frac{1}{2})} y^{-1} e^{-y^2/2} \frac{d}{dy} y^2 = \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} e^{-y^2/2}$$

This is for $y \ge 0$. The range is $[0, \infty)$.

3. I roll two tetrahedral (four-sided) dice. Let X be the number of dice that show an odd number, Y the number that show an even number. So both X and Y only take on the values $0, 1, 2$.

	Y=0	Y=1	Y=2
X=0	0	0	1/4
X=1	0	1/2	0
X=2	1/4	0	0

(a) Find the joint probability mass function (pmf) for X, Y. You can give your answer in the form of a table. **Solution:** Note that $X + Y = 2$. So the joint pmf is only nonzero for x, y that sum to 2.

(b) Compute $P(Y \geq X)$. **Solution:** $1/2 + 1/4 = 3/4$.

(c) Are X and Y independent? Justify your answer. Note: you can do (c) without doing (a) or (b). **Solution:** No. For example, $P(X = 0, Y = 0) = 0$, but $P(X = 0) > 0$ and $P(Y > 0)$. So the joint pmd is not the product of the marginal pmf's of X and Y.

4. Let X and Y be independent continuous random variables. X has a uniform distribution on $[0, 1]$ and Y has an exponential distribution with $E[Y] = 1$. Let $Z = Y - X$. Compute $P(Z \geq 0)$. **Solution:** The joint pdf is e^{-y} for $0 \leq x \leq 1$ and $y \geq 0$. So after drawing a picture to set up the limits we find

$$P(Z \geq 0) \quad = \quad P(Y \geq X) = \int_0^1 \left(\int_x^\infty e^{-y} dy \right) dx = \int_0^1 e^{-x}\, dx = 1 - e^{-1}$$

5. I roll a fair die until I get a 1 or a 2. Let X be the number of rolls this takes. Then I continue rolling it until I get that same number a second time. Let N be the total number of rolls.

(a) Find the mean and variance of X.

Solution: X is a geometric RV with $p = 1/3$. So its mean is $1/p = 3$ and its variance is $(1 - p)/p^2 = 6$.

(b) Find the mean and variance of N. Hint : you can write N as $X + Y$ where X and Y are independent, relatively simple RV's.

Solution: Let Y be the number of additional flips it takes to get 1 or 2 for the second time. So $N = X + Y$ and Y is geometric with $p = 1/6$. So $E[Y] = 6$ and $var(Y) = 30$. So $E[N] = 3+6 = 9$. X and Y are independent, so $var(N) = var(X) + var(Y) = 6 + 30 = 36$.

6. The joint probability density function (pdf) of two continuous random variables X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 4\,e^{-2x-y}, & \text{if } x \geq 0, y \geq 0, y \geq 2x \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal pdf of X. **Solution:** First note that the range of X is $x \geq 0$. For $x \geq 0$ the marginal is found by integrating out y:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) = \int_{2x}^{\infty} 4\,e^{-2x-y}\,dy = e^{-2x} \int_{2x}^{\infty} 4\,e^{-y}\,dy = 4e^{-4x}$$

So

$$f_X(x) = \begin{cases} 4\,e^{-4x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) Find $P(Y \geq 1 | X = x)$, i.e., the probability that $Y \geq 1$ given that $X = x$. **Solution:** For $x \geq 0$, the conditional pdf of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{4\,e^{-2x-y} 1(x \geq 0, y \geq 0, y \geq 2x)}{4e^{-4x} 1(x \geq 0)}$$

We can only condition on $X = x$ when $x \geq 0$, so we can drop the conditions $x \geq 0$ in the above. So

$$f_{Y|X}(y|x) = \frac{4\,e^{-2x-y} 1(y \geq 0, y \geq 2x)}{4e^{-4x}} 4\,e^{2x-y} 1(y \geq 0, y \geq 2x)$$

We need to integrate this with respect to y from $y = 1$ to $y = \infty$. If $x \leq 1/2$, the integral runs from 1 to ∞ and we get

$$P(Y \geq 1 | X = x) = \int_1^{\infty} 4\,e^{2x-y}\,dy = \int_1^{\infty} 4\,e^{2x-y}\,dy = e^{2x} \int_1^{\infty} 4\,e^{-y}\,dy = e^{2x-1}$$

If $x \geq 1/2$ then Y is always at least 1. So

$$P(Y \geq 1 | X = x) = \begin{cases} e^{2x-1}, & \text{if } x \leq 1/2 \\ 1, & \text{if } x \geq 1/2 \end{cases}$$

7. Let X and Y be independent random variables. Both are exponential with $\lambda = 2$. Let

$$U = \frac{1}{2}(X + Y), \qquad W = Y^2$$

(a) What is the range of (U, W)? **Solution:** Solving for x, y in terms of u, w we find $x = 2u - \sqrt{w}, y = \sqrt{w}$. The range of X, Y is the upper right quadrant. We look at what happens to the two boundaries of this quadrant. When $y = 0, x \geq 0$, $u = x/2, w = 0$. So this is just the positive horizontal axis. When $x = 0, y \geq 0$, $u = y/2, w = y^2$. So the points (u, w) lie on the parabola $w = 4u^2$. So the range is $\{(u, w) : u \geq 0, w \leq 4u^2$. Geometrically this is the part of the upper right quadratic bounded by the horizontal axis below and the parabola above.

(b) Find the joint pdf for U, W. **Solution:** We have $f_{X,Y}(x, y) = 4e^{-2x-2y}$. The Jacobian works out to $w^{-1/2}$. So

$$f_{U,W}(u, w) = 4e^{-4u}w^{-1/2}$$

on the range given in (a).

8. Let X_j be a sequence of independent, identically distributed random variables. Their common density is

$$f(x) = 4xe^{-2x}, x \geq 0$$

(It is zero for $x < 0$.) Let

$$\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$$

(a) Find the mean and variance of $\overline{X}_n$. Hint: this density is in our catalog.
Solution: The X_j have a gamma distribution with $w = 2$ and $\lambda = 2$. So $E[X_j] = 1$ and $var(X_j) = 1/2$. So the mean of $\overline{X}_n$ is 1 and its variance is $\frac{1}{2n}$.

(b) For $n = 1000$, the probability that $\overline{X}_{1000}$ is in $[1, 1.1]$ is approximately given by

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Find a and b.
Solution:

$$\begin{aligned} P(1 \leq \overline{X}_{1000} \leq 1.1) &= P(0 \leq \frac{\overline{X}_{1000} - 1}{\frac{1}{\sqrt{2n}}} \leq \frac{1.1 - 1}{\frac{1}{\sqrt{2n}}}) \\ &= P(0 \leq Z \leq 0.1\sqrt{2n}) \end{aligned}$$

So $a = 0$ and $b = 0.1\sqrt{2n} = 2\sqrt{5}$.

9. The time it takes to solve a problem is exponentially distributed with $\lambda = 1/2$. Two students named Fred and Wilma begin working on the problem at the same time. (Assume that the students are working independently).

(a) Find the probability that at least one of the students has solved the problem at the end of 2 minutes.

Solution: For one of them, the probability of not having solved it by two minutes is

$$\int_2^\infty \frac{1}{2} e^{-x/2}\, dx = e^{-1}$$

Now

$$\begin{aligned} P(at\,least\,one\,solution) &= 1 - P(no\,solution) \\ &= 1 - P(Fred\,no\,solution)P(Wilma\,no\,solution) \\ &= 1 - e^{-2} \end{aligned}$$

(b) Find the probability that Fred takes at least twice as long as Wilma to solve the problem.

Solution: We have

$$\begin{aligned} P(Y \geq 2X) &= \int_0^\infty \left[\int_{2x}^\infty \frac{1}{4} e^{-(x+y)/2} dy \right] dx = \frac{1}{4} \int_0^\infty e^{-x/2} \left[\int_{2x}^\infty e^{-y/2} dy \right] dx \\ &= \frac{1}{2} \int_0^\infty e^{-x/2} e^{-x} dx = \frac{1}{2} \int_0^\infty e^{-3x/2} dx = \frac{1}{3} \end{aligned}$$

10. A mystery random number generator produces random real numbers that are uniformly distributed between a and $a+1$. I don't know what a is. (a is not random, it is just unknown.) I call the random number generator n times and let $\overline{X}_n$ be the average of the n numbers I get. (So $\overline{X}_n$ is the sum of the n numbers divided by n.) As n goes to ∞, $\overline{X}_n$ converges to $a + 1/2$. Find the smallest n so that the probability that $\overline{X}_n$ is within 0.01 of $a + 1/2$ is approximately 0.05. For a standard normal RV Z, we have $P(Z \leq -2.576) = 0.005$,
$P(Z \leq -2.326) = 0.01$, $P(Z \leq -1.960) = 0.025$, $P(Z \leq -1.649) = 0.05$.

Solution: Let X_i be the random numbers with $i = 1, 2, \cdots n$. This is an i.i.d. sequence with mean $a + 1/2$ and variance (from the formula sheet) 1/12. So we standardize $\overline{X}_n$ by letting

$$Z_n = \frac{\overline{X}_n - (a + 1/2)}{\sqrt{1/12n}}$$

11. X and Y are discrete random variables with joint pmf

$$f_{X,Y}(x, y) = \frac{\lambda^{x+y} \, e^{-2\lambda}}{x! \, y!}$$

where x and y both take on the values $0, 1, 2, 3, \cdots$, and λ is a positive parameter.
(a) Find the marginal pmf's of X and Y. **Solution:**

$$\begin{aligned} f_X(x) &= \sum_y f_{X,Y}(x, y) = \sum_{y=0}^{\infty} \frac{\lambda^{x+y} \, e^{-2\lambda}}{x! \, y!} = e^{-2\lambda} \frac{\lambda^x}{x!} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} \\ &= e^{-2\lambda} \frac{\lambda^x}{x!} \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} == e^{-2\lambda} \frac{\lambda^x}{x!} e^{\lambda} = e^{-\lambda} \frac{\lambda^x}{x!} \end{aligned}$$

This is a Poisson distribution with parameter λ. Similarly Y has a Poisson distribution with parameter λ.
(b) Find $E[XY]$. **Solution:** Note that the joint pmf is the product of the marginal pdf's of X and Y. So X and Y are independent. So $E[XY] = E[X]E[Y]$. The mean of a Poisson RV is λ, so $E[XY] = \lambda^2$.
(c) Let $Z = X + Y$. Find $E[Z|X = x]$. **Solution:** The conditional pdf of Z given $X = x$ is

$$f_{Z|X}(z|x) = \frac{f_{Z,X}(z, x)}{f_X(x)}$$

We know f_X already, but we need to find $f_{Z,X}$. First note that $P(Z = z, X = x) = 0$ if $z < x$. For $z \geq x$,

$$f_{Z,X} = P(Z = z, X = x) = P(X = x, Y = z - x) = \frac{\lambda^z \, e^{-2\lambda}}{x! \, (z - x)!}$$

So

$$f_{Z|X}(z|x) = \frac{\frac{\lambda^z \, e^{-2\lambda}}{x!\,(z-x)!}}{\frac{\lambda^x e^{-\lambda}}{x!}} = \frac{\lambda^{z-x} \, e^{-\lambda}}{(z-x)!}$$

This is the pdf of a Possion RV shifted by x. So it has mean x plus the mean of the Poisson which is λ. So $E[Z|X = x] = x + \lambda$.

(d) Let $Z = X + Y$. Find the moment generating function of Z. Z is in our catalog. What is it? **Solution:**

$$M_Z(t) = M_X(t)M_Y(t) = \exp(\lambda(e^t - 1)) \exp(\lambda(e^t - 1)) = \exp(2\lambda(e^t - 1))$$

This is the mgf of a Possion RV with parameter 2λ.

12. I roll a fair six-sided die until I get an even number. Call it N. Then I roll it some more until I get a number at least as big as N. Let X be the total number of rolls. Find the mean of X.

Solution: Let Y be the number of rolls to get the first even number N. Let Z be the number of additional rolls to get something $\geq N$. So $X = Y+Z$. Y is geometric with $p = 1/2$. So $E[Y] = 2$. To compute $E[Z]$ we condition on N:

$$E[Z] = \sum_{n=2,4,6} E[Z|N = n]P(N = n)$$

$P(N = n)$ is just $1/3$. Consider being given $N = 2$. Then Z is the number of rolls to get $2, 3, 4, 5, 6$. This is geometric with $p = 5/6$. Similarly, given $N = 4$, Z is geometric with $p = 3/6$. And given $N = 6$, Z is geometric with $p = 1/6$.

$$E[Z] = \frac{1}{3}(\frac{6}{5} + \frac{6}{3} + \frac{6}{1}) = \frac{46}{15}$$

Thus $E[X] = 2 + \frac{46}{15} = \frac{76}{15}$.

13. A continuous random variable Z has range equal to the entire real line with pdf $f_Z(z) = e^{-|z|}/2$.

(a) Find the moment generating function (mgf) of Z. **Solution:** Note that $|x|$ equal x when $x > 0$ and equals $-x$ when $x < 0$. So

$$M_Z(t) = \int_{-\infty}^{\infty} e^{tz} \, \frac{1}{2} e^{-|z|} \, dz$$

$$
\begin{aligned}
&= \int_0^\infty e^{tz}\,\frac{1}{2}e^{-z}\,dz + \int_0^\infty e^{tz}\,\frac{1}{2}e^{z}\,dz \\
&= \frac{1}{2}\int_0^\infty e^{(t-1)z}\,dz + \frac{1}{2}\int_0^\infty e^{(t+1)z}\,dz \\
&= \frac{1}{2}\frac{1}{1-t} + \frac{1}{2}\frac{1}{1+t} = \frac{1}{1-t^2}
\end{aligned}
$$

(b) Let X and Y be independent and identically distributed random variables. They have the same mgf, call it $M(t)$. Let $W = X - Y$. Show that the mgf of W satifies $M_W(t) = M(t)M(-t)$. **Solution:**

$$M_W(t) = E[e^{tW}] = E[e^{t(X-Y)}] = E[e^{tX}e^{-tY}] = E[e^{tX}]E[e^{-tY}] = M_X(t)M_Y(-t)$$

(c) If we take X and Y to be independent and identically distributed with one of the continuous pdf's in our catalog, then $W = X - Y$ will have the same pdf as Z. For which pdf in the catalog is this true? You should justify your answer. Hint: $1 - t^2 = (1-t)(1+t)$. **Solution:** The mgf of Z is $1/(1-t^2) = 1/[(1-t)(1+t)]$. So part (b) says we should look for a pdf with mgf $1/(1-t)$. The exponential distribution with $\lambda = 1$ has this mgf.