7. Renormalization Theory for Ising-like Spin Systems

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Introduction

Most of the applications of the renormalization theory, as proposed by Wilson (Wilson, 1971a, b; Wilson and Fisher, 1972; Wilson and Kogut,

1974) for many-body systems, concern lattices with classical (i.e. continuous), n-component spin variables, where $\varepsilon = 4 - d$ (d is the dimensionality of the lattice) has to be used as an expansion parameter for physical quantities. Here we shall be concerned with spins which can assume only the values ± 1. Such a limitation, though certainly not essential (e.g. Nihat Berker, 1975) for the method outlined here, greatly simplifies the introduction of renormalization transformations and the actual calculation of the free energy, including its critical singularities, from these transformations.

The essence of the renormalization transformation approach to systems with many degrees of freedom is a stepwise evaluation of the free energy. Starting with the partition function of N spins s_i on a lattice, as the sum over all configurations $\{s\}$, this summation is carried out under the restriction that the configurations $\{s\}$ match with a given cell spin configuration $\{s'\}$. Here the N' cell spins $s'_{i'}$ assume the values ± 1 and are located on a lattice which is isomorphic with the original site lattice. The cell spin lattice is smaller by a scale factor $N'/N = l^{-d}$, where l is the linear dimension of the cell in units of the site lattice. The result can be written as the free energy of the cell spin configurations, which in turn defines uniquely a Hamiltonian for the cell spins.

Instead of summing over all the cell spins to obtain the free energy of the original site system, one considers the transition of the Hamiltonian describing the site system to the Hamiltonian describing the cell system. This transition obviously is a transformation in the space of Hamiltonian parameters for the site lattice: the renormalization transformation. Since the cell lattice is isomorphic with the original lattice, this transformation in the space of Hamiltonian parameters can be iterated and one is most interested in the cases where the sequence of Hamiltonians approaches a non-trivial limit (fixed point).

For the usefulness of a renormalization transformation it is essential that it is regular in the domain of interest in the space of Hamiltonian parameters. In that case the free energy of the cell system and the site system must exhibit the same singularities and one might call the two systems, which are connected by the transformation, thermodynamically equivalent. In general it is quite easy to write down a renormalization transformation, but hard to assess whether it is regular, particularly near its non-trivial fixed points. A special criterion that has to be satisfied is that critical systems are carried over into critical systems. For a certain class of renormalization transformations, those where the cell spins and the site spins are linearly related, this criterion is satisfied automatically.

We shall, however, pay little attention to these "linear" transformations but concentrate on transformations where the relation between the cell and site spins is non-linear. One of the main points, which now becomes much

more cumbersome, is to discuss the behaviour of the correlation functions, which will be done in Section III. But apart from considerations about the correlation functions one can indicate the usefulness of the renormalization theory for critical phenomena by considering the fluctuations.

One knows from the work of Landau that the critical singularities are caused by long wavelength fluctuations. A long wavelength fluctuation will manifest itself more in the configuration of cell spins than in the site spin configurations compatible therewith. One might say that by leaving the cell spin variables in the problem the important fluctuations are still kept in the system. Therefore the transformation, which involves only a sum over site spins for a given cell spin configuration, does not involve the "dangerous" fluctuations and will thus be regular.

The cell spin Hamiltonian may even be seen as an expression for the energy of the fluctuations of the site spins. The fluctuation energy is expressed with respect to a mean value, which is going to play a fundamental rôle in our treatment. This mean value is a kind of background energy with respect to which fluctuations are measured energetically. On repeating the renormalization transformation the free energy is gradually expressed as a sum over this background energy. Near the fixed point this sum builds up the typical critical singularities.

The theory for the correlation functions is set up in a similar way. In terms of fluctuations the argument runs parallel; when a spin pair correlation function is long-ranged, this is a result of the long wavelength fluctuations. It turns out that by applying the renormalization transformation repeatedly, the full correlation function is expressed in terms of a regular short-ranged part, in some analogy with the Ornstein–Zernike theory.

In Section I we start with definitions and introduce the basic concepts such as cell spins, fixed points, eigenvectors, critical exponents and scaling fields. In Section II the thermodynamic implications of the renormalization theory, including the spontaneous magnetization, are discussed, and similarly in Section III the implications of the renormalization transformations for the correlation functions. In Section IV a number of computational methods are introduced and in Section V a survey of the results obtained so far is given. The paper closes with a discussion, Section VI, of several questions that have been raised in the preceding sections.

I. Definitions

We consider a lattice with a spin $s_i = \pm 1$ on every site. The subscript i labels the lattice sites; for the case of d-dimensional lattices i usually stands for d labels $i_1, \ldots, i_d$ to mark the coordinate in every direction respectively.

Let a be a subset of sites and define s_a as

$$s_a = \prod_{i \in a} s_i \tag{1.1}$$

The most general Hamiltonian describing an Ising-like system can then be written as

$$\mathcal{H}(s) = \sum_a K_a s_a \tag{1.2}$$

where the usual Boltzmann factor $-\beta = -1/k_B T$ has already been absorbed in the Hamiltonian, i.e. the "actual" interaction parameters are $J_a = K_a/k_B T$. For future convenience the total energy is shifted in such a way that

$$\sum_{\{s\}} \mathcal{H}(s) = 0 \tag{1.3}$$

where the symbol $\Sigma_{\{s\}}$ will always mean a summation over all the spin configurations. Also a in Σ_a of (1.2) runs through all non-empty subsets of sites.

The interaction parameters K_a can be formally obtained from $\mathcal{H}(s)$ via the relation

$$K_a = 2^{-N} \sum_{\{s\}} s_a \mathcal{H}(s) \tag{1.4}$$

where N stands for the number of lattice sites. Equation (1.4), recovering the interaction parameters K_a from the Hamiltonian $\mathcal{H}(s)$, is of importance for actual numerical calculations.

In practice we shall mostly deal with homogeneous Hamiltonians (1.2), i.e. Hamiltonians with the following property: let α be the class of all subsets a of sites which can be identified by a symmetry operation of the lattice; α may e.g. stand for "single site" (K_α then being a magnetic field), "nearest neighbour pair", etc. We shall restrict ourselves to systems where all K_a of $a \in \alpha$ have the same value K_α. For these homogeneous Hamiltonians, (1.2) can be rewritten as

$$\mathcal{H}(s) = \sum_a K_a s_a = \sum_\alpha K_\alpha \sum_{a \in \alpha} s_a \tag{1.5}$$

Obviously (1.5) can be written in a less formal way as

$$\mathcal{H}(s) = H \sum_i s_i + K \sum_{\langle i,j \rangle} s_i s_j + \ldots \tag{1.6}$$

where the symbol $\langle i,j \rangle$ as usual means that the sites i and j are nearest neighbours. The restriction to homogeneous systems, however, is by no means necessary. In fact this approach to renormalization theory for Ising-like spin systems can even be applied to systems with random interactions (Harris and Lubensky, 1974).

Specific examples of homogeneous systems (1.5) for which exact solutions are known are e.g.:

(a) "Pure" Ising systems defined by the Hamiltonian

$$\mathcal{H}(s) = K \sum_{\langle i,j \rangle} s_i s_j \tag{1.7}$$

for the linear and 2-dimensional lattices (Onsager, 1944).

(b) The triple-spin interaction model, which is defined on the 2-dimensional triangular lattice by the Hamiltonian (Baxter and Wu, 1973)

$$\mathcal{H}(s) = K \sum_{\langle i,j,k \rangle} s_i s_j s_k \tag{1.8}$$

where the symbol $\langle i, j, k \rangle$ stands for the sites of the vertices of any triangle.

(c) The symmetric 8-vertex model (or Baxter (1971, 1972) model), which can also be expressed as an Ising-like system on a quadratic lattice by the following Hamiltonian ($i = (i_1, i_2)$ is the site position):

$$\mathcal{H}(s) = K \sum_{i_1, i_2} [s_{i_1, i_2} s_{i_1+1, i_2+1} + s_{i_1+1, i_2} s_{i_1, i_2+1}] + \lambda \sum s_{i_1, i_2} s_{i_1+1, i_2+1} s_{i_1+1, i_2} s_{i_1, i_2+1} \tag{1.9}$$

Note that, by the absence of a magnetic field, the Hamiltonians (1.7) and (1.9) both only contain terms which are even in the spin variables, whereas (1.8) only contains a term which is odd in the spin variables.

Now, returning to the general theory, consider a set of N' so-called cell spins $s_{i'} = \pm 1$ (the reason for this nomenclature will soon become clear). The index i' refers to the i'-th cell of a lattice isomorphic to the original lattice. All quantities pertaining to the cell spin system will be labelled with primes. Furthermore, let $P(s', s)$ be a weight factor, which depends on the cell spin and site spin configurations $\{s'\}$ and $\{s\}$, with the properties

$$P(s', s) \geqslant 0, \qquad \forall s', s \tag{1.10a}$$

and

$$\sum_{\{s'\}} P(s', s) = 1 \tag{1.10b}$$

We now define a Hamiltonian $\mathcal{H}'(s')$ for the cell spin system by

$$\exp[G + \mathcal{H}'(s')] = \sum_{\{s\}} P(s', s) \exp \mathcal{H}(s) \tag{1.11}$$

and

$$\sum_{\{s'\}} \mathcal{H}'(s') = 0 \tag{1.12}$$

G is independent of the cell spin configurations $\{s'\}$ and is defined by imposing the condition (1.12) on the cell spin Hamiltonian $\mathcal{H}'(s')$.

The restriction (1.10a) has been imposed on $P(s', s)$ in order to guarantee that the right-hand side of (1.11) is positive and that thus G and $\mathcal{H}'(s')$ are real quantities. As (1.10a) is sufficient but by no means necessary, this condition is sometimes relaxed in some calculations.

The second condition, (1.10b), leads to the following important relationship between the free energy F' of the cell system and the free energy F of the site system:

$$G + F' = \ln \sum_{\{s'\}} \exp [G + \mathcal{H}'(s')] = \ln \sum_{\{s'\}} \sum_{\{s\}} P(s', s) \exp \mathcal{H}(s)$$
$$= \ln \sum_{\{s\}} \exp \mathcal{H}(s) = F \tag{1.13}$$

A third condition will be imposed on $P(s', s)$ as well, in order to guarantee that the cell spin Hamiltonian $\mathcal{H}'(s')$ has the same symmetries as the site spin Hamiltonian $\mathcal{H}(s)$. To elucidate this we decompose $\mathcal{H}'(s')$ by means of (1.4) as

$$\mathcal{H}'(s') = \sum_{a'} K'_{a'} s'_{a'} \tag{1.14}$$

where a' runs through all non-empty subsets of cells. Then we require $P(s', s)$ such that for subsets a' of a certain class α, the cell interaction parameters $K'_{a'}$ are all equal to a certain value K'_α, i.e. $\mathcal{H}'(s')$ can again be written as

$$\mathcal{H}'(s') = \sum_{\alpha} K'_\alpha \sum_{a' \in \alpha} s'_{a'} \tag{1.15}$$

Note that we do not use primes on the classes of sets of cells; they are superfluous since e.g. the class of nearest neighbours is a definition which is independent of whether we are working in the cell or in the site system.

Now consider F' and F as functions of the interaction parameters K'_α and K_α (abbreviated to K' and K). In the thermodynamic limit (no long-range forces present) F' and F assume the form $F' = N' f(K')$ and $F = Nf(K)$, with in both cases the *same* function f. G also becomes an extensive function $G = Ng(K)$ in the thermodynamic limit with $N/N' = l^d$ constant (d = dimensionality of the system). Inserting these relations into (1.13) leads to the following basic renormalization relation for the free energy per site:

$$f(K) = g(K) + l^{-d} f(K') \tag{1.16}$$

Obviously l is also the linear distance between the cells measured in units of the site lattice.

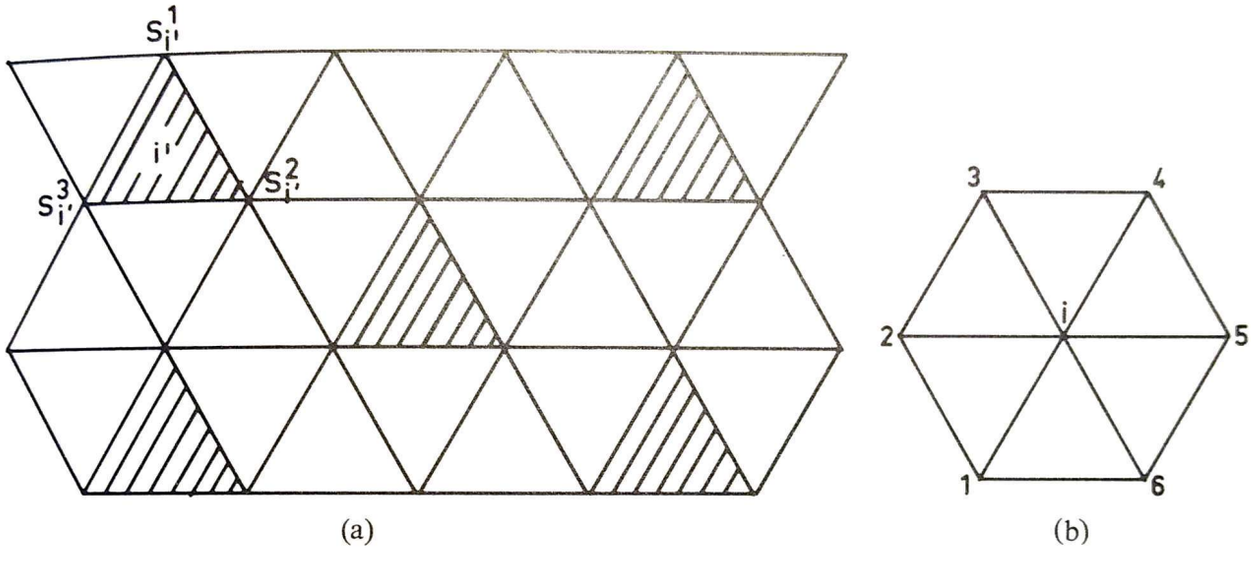
The relation (1.11) is called a renormalization transformation. The three above-mentioned restrictions on $P(s', s)$ still allow an enormous variety in choice for $P(s', s)$, each leading to a different renormalization transformation.

A trivial choice would be to take $P(s', s)$ independent of s', which in view of (1.10b) leads to

$$P(s', s) = 2^{-N'} \tag{1.17}$$

This makes the right-hand side of (1.11) independent of s' and therefore $\mathscr{H}'(s') = 0$ and $G = F$.

As an example, to which we shall return repeatedly in Sections IV and V and for which extensive calculations have been carried out, we give here a more interesting choice for a 2-dimensional triangular lattice (Niemeijer and van Leeuwen, 1973, 1974). Consider a triangular lattice where cell spins have been placed in the centre of three sites as indicated in Fig. 1a.



(a) (b)

FIG. 1

Note that the cell spins are again located on a triangular lattice. Now define the weight-factor

$$P(s', s) = \prod_{i'} \tfrac{1}{2}[1 + s'_{i'}(s^1_{i'} + s^2_{i'} + s^3_{i'} - s^1_{i'}s^2_{i'}s^3_{i'})/2] \tag{1.18}$$

with $s^1_{i'}$, $s^2_{i'}$ and $s^3_{i'}$ denoting the sites in cell i'. One easily verifies that (1.18) fulfils the conditions (1.10a and b). The symmetry condition on this particular $P(s', s)$ is not quite fulfilled, but this can be remedied by averaging over all six triangle positions $i' = (i, 1, 2)$ etc. indicated in Fig. 1b. The choice (1.18) is special, since $P(s', s)$ is either 0 or 1. $P(s', s)$ equals 1 when

$$s'_{i'} = \text{sign}\,(s^1_{i'} + s^2_{i'} + s^3_{i'}) \tag{1.19}$$

and is zero otherwise. With the $P(s', s)$ given by (1.18) the transformation

(1.11) can be interpreted differently. For a given value of $s'_{i'}$ the three neighbouring site spins can still be in four configurations, labelled by $\sigma_{i'}$:

$$\begin{aligned}
\sigma_{i'} = 0: &\quad s^1_{i'} = s^2_{i'} = s^3_{i'} = s'_{i'},\\
\sigma_{i'} = 1: &\quad -s^1_{i'} = s^2_{i'} = s^3_{i'} = s'_{i'},\\
\sigma_{i'} = 2: &\quad s^1_{i'} = -s^2_{i'} = s^3_{i'} = s'_{i'},\\
\sigma_{i'} = 3: &\quad s^1_{i'} = s^2_{i'} = -s^3_{i'} = s'_{i'}.
\end{aligned} \tag{1.20}$$

So effectively the summation in (1.11) over $\{s\}$ runs over the possible configurations $\{\sigma\}$ and one may write (1.11) as

$$\exp\left[G + \mathcal{H}'(s')\right] = \sum_{\{\sigma\}} \exp \mathcal{H}(s). \tag{1.21}$$

A choice as (1.18) has computational advantages in limiting the number of configurations to be summed over. The relation between cell and site spins as implied by (1.18) may be compared to the election of delegates in a district system with only two parties present. The site spins (voters) of a cell (district) determine the cell spin (delegate) by a majority rule. (We note here in passing that the square of the transformation (1.18) is not equal to the transformation based on taking a "super cell" of three cells as the basic unit, just as the minority may elect the president by an advantageous distribution of the votes over the districts. In this respect (1.18) is an example of a non-linear weight-factor which we shall discuss in Section III.)

Returning to (1.11), we view this as a transformation from the interaction parameters K_α to the renormalized interaction parameters K'_α, in short

$$K' = K'(K). \tag{1.22}$$

One can consider $G(K)$ as a special K'_ϕ with ϕ corresponding to the empty set. Equations (1.22) will be taken as the renormalization equations, which together with (1.16) determine the free energy $f(K)$. The strategy of the renormalization theory will be to obtain the singularities of the free energy $f(K)$ from the regular functions $g(K)$ and $K'(K)$. There is, however, no *a priori* guarantee that $g(K)$ and $K'(K)$ will be regular in the region of interest, since the choice (1.17) shows that $g(K) = f(K)$ and $g(K)$ then has all the singularities of $f(K)$. It is part of the renormalization problem to choose $P(s', s)$ in such a way that $g(K)$ and $K'(K)$ are regular in the relevant domain.

We now introduce the important concept of a fixed point of the equations (1.22). A fixed point K^* is a set K^*_α such that

$$K'_\alpha(K^*) = K^*_\alpha. \tag{1.23}$$

Of course the transformation (1.22) can in general have several fixed points,

but not every one of them will have physical significance for critical phenomena; e.g. one trivial fixed point is the set K_α^* with

$$K_\alpha^* = 0 \tag{1.24}$$

corresponding to the infinite temperature limit of any spin system (since the parameter $-\beta = -1/k_B T$ has been absorbed in the interaction parameters).

It is one of the basic requirements of the renormalization approach that, for the determination of the critical exponents of a class of Ising systems, which have the same critical behaviour, the K'_α are regular functions of the K_α near the non-trivial fixed points. For the determination of the free energy of a specific spin system we shall need $g(K)$ and the transformation (1.22) to be regular in a much larger region of the parameter space.

For later results we shall need the linearized form of the transformation (1.22), i.e. the matrix

$$\partial K'_\alpha / \partial K_\beta = T_{\alpha\beta}. \tag{1.25}$$

A very special rôle is played by this linearized transformation in the fixed point (the fact that it is linearized at the fixed point will be denoted by an asterisk)

$$(\partial K'_\alpha / \partial K_\beta)_{K^*} = T^*_{\alpha\beta}. \tag{1.26}$$

So in the neighbourhood of K^* the transformation can be written as:

$$K'_\alpha - K^*_\alpha = \sum_\beta T^*_{\alpha\beta}(K_\beta - K^*_\alpha). \tag{1.27}$$

The matrix $T^*_{\alpha\beta}$ is not symmetric, so *a priori* one cannot be sure that its eigenvalues, which will turn out to be directly related to the critical exponents, are real. A general feature of $T^*_{\alpha\beta}$ in many applications is, that its elements below the diagonal are usually much smaller than those above the diagonal.

It is convenient to introduce a type of normal coordinates in which the transformation (1.27) takes a simple form. Let $T^*_{\alpha\beta}$ have eigenvalues λ_i with associated left eigenvectors φ^i_α:

$$\sum_\alpha \varphi^i_\alpha T^*_{\alpha\beta} = \lambda_i \varphi^i_\beta. \tag{1.28}$$

Then construct "normal" coordinates u_i as

$$u_i = \sum_\alpha \varphi^i_\alpha (K_\alpha - K^*_\alpha) \tag{1.29}$$

such that the u_i according to (1.27) and (1.28) transform as

$$u'_i = \sum_\alpha \varphi^i_\alpha (K'_\alpha - K^*_\alpha) = \lambda_i \sum_\beta \varphi^i_\beta (K_\beta - K^*_\beta) = \lambda_i u_i \tag{1.30}$$

One may extend the definition of u_i beyond the linear regime in such a way

that (1.30) also holds for finite u_i. For that purpose (1.27) has to be expanded further as

$$K'_\alpha - K^*_\alpha = \sum_\alpha T^*_{\alpha\beta}(K_\beta - K^*_\beta) + \frac{1}{2!}\sum_{\beta,\gamma} T^*_{\alpha,\beta\gamma}(K_\beta - K^*_\beta)(K_\gamma - K^*_\gamma)\ldots \tag{1.31}$$

as well as (1.29)

$$u_i = \sum_\alpha \varphi^i_\alpha(K_\alpha - K^*_\alpha) + \frac{1}{2!}\sum_{\alpha\beta} \varphi^i_{\alpha\beta}(K_\alpha - K^*_\alpha)(K_\beta - K^*_\beta)\ldots \tag{1.32}$$

where $T^*_{\alpha,\beta\gamma} = (\partial^2 K_\alpha/\partial K_\beta \partial K_\gamma)_{K^*}$. In order to ensure that

$$u'_i = \lambda_i u_i \tag{1.33}$$

also holds to second order, one should relate $\varphi^i_{\alpha\beta}$ to $T^*_{\alpha,\beta\gamma}$ as the solutions of the linear set of equations

$$\sum_{\mu,\nu} \varphi^i_{\mu\nu}[\lambda_i \delta_{\mu\alpha}\delta_{\nu\beta} - T^*_{\mu\alpha}T^*_{\nu\beta}] = \sum_\mu \varphi^i_\mu T^*_{\mu,\alpha\beta}. \tag{1.34}$$

The solutions of (1.34) are

$$\varphi^i_{\mu\nu} = \sum_{j,k} \frac{1}{\lambda_i - \lambda_j\lambda_k} \sum_{\gamma\alpha\beta} \varphi^i_\gamma T^*_{\gamma,\alpha\beta}\psi^j_\alpha \psi^k_\beta \varphi^j_\mu \varphi^k_\nu. \tag{1.35}$$

(where ψ^i_α is the αth component of the right eigenvector of $T^*_{\alpha\beta}$ with eigenvalue λ_i).

To find the u_i as functions of K, it is for practical purposes more convenient to use the relation

$$\lambda_i u_i(K) = u_i(K') \text{ or } \lambda_i^n u_i(K) = u_i(K^{(n)}) \tag{1.36}$$

in which $K^{(n)}$ is the image of K after n transformations. By choosing K in the neighbourhood of the fixed point K^* one finds (except under special circumstances) values of $u_i(K^{(n)})$ further away from K^*. The set $\{u_i(K)\}$ may be considered as the curvilinear coordinates adapted to the transformation (Wegner's scaling fields, Wegner, 1972). Note that the u_i are defined up to a constant factor, since the eigenvectors are only defined up to a constant factor. It should also be pointed out that both procedures fail in case of an eigenvalue $\lambda = 1$ (a so-called marginal eigenvalue).

II. General Theory

This Section will be devoted to the consequences of the basic formula (1.16), viz.

$$f(K) = g(K) + l^{-d} f(K')$$

which in terms of the scaling fields u_i can be written as:

$$f(u_1, u_2, \ldots) = g(u_1, u_2, \ldots) + l^{-d} f(\lambda_1 u_1, \lambda_2 u_2, \ldots). \tag{2.1}$$

Our basic assumption is that the scaling fields u_i are regular functions of the interaction parameters K_α and that $g(u_1, u_2, \ldots)$ is a regular function of the scaling fields u_i.

If a field u_i scales under the renormalization transformation with a factor $|\lambda_i| > 1$ (<1) it is called relevant (irrelevant). In the special case that $\lambda_i = 1$ the field is called marginal. In order to appreciate further the notion of scaling fields we point out that, obviously, the values of the relevant fields increase upon performing a transformation, whereas the irrelevant fields decrease. The fixed point itself, as defined in (1.23) corresponds to all $u_i = 0$. The set of points which ultimately end up in the fixed point is called the domain of attraction; this set of points forms a hypersurface (Fig. 2) in the parameter space of the interactions. If there are n interactions and m relevant variables, the dimension of this hyper surface is $n - m$. For reasons which will become clear later on, the domain of attraction is also called the *surface of criticality*. If e.g. u_1 and u_2 are relevant fields, the surface of criticality is defined by $u_1 = u_2 = 0$.

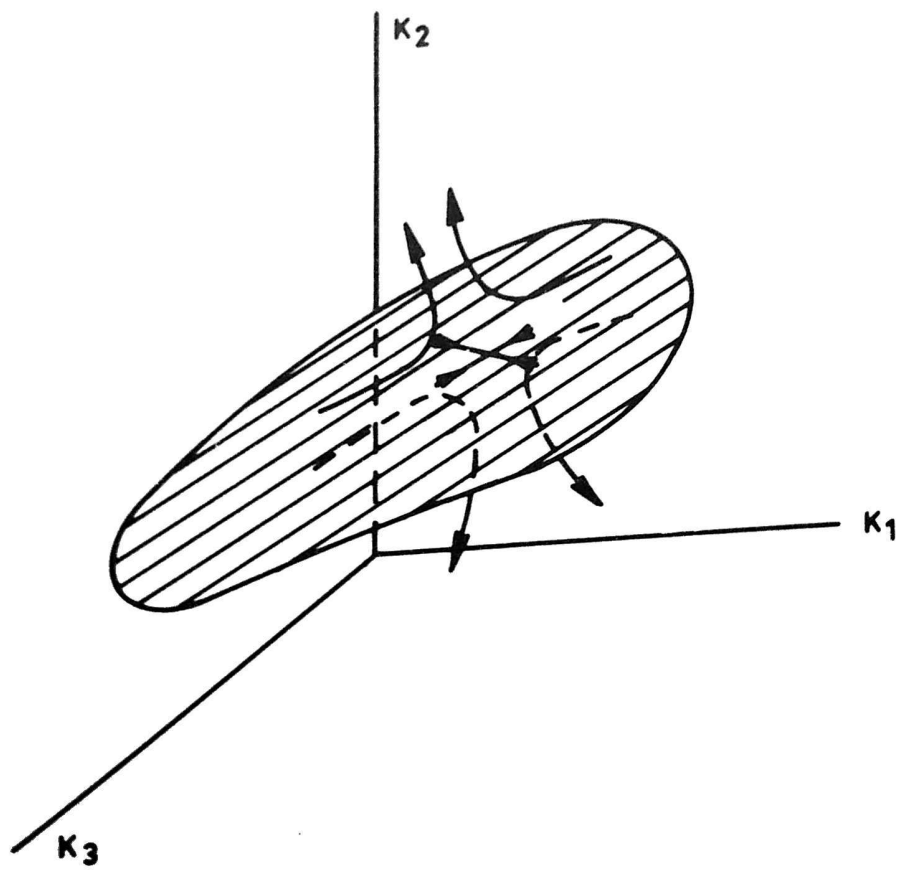


FIG. 2

Another general distinction between the scaling fields can be made on the basis of symmetry. Each $\mathcal{H}(s)$ is made up of an even part, which is invariant under a flip of all spins, and an odd part, which changes sign under a spin flip. Now, if all $P(s', s)$ are chosen such that they are invariant under a simultaneous flip of s' and s, then it follows that the subspace of the even interactions of $\mathcal{H}(s)$ is invariant under the transformation. Thence it follows that the subspace of the even part of $\mathcal{H}(s)$ is invariant under the transformation, and as a consequence, in most cases, such as (1.9) and (1.7), fixed points are found in the subspace of even interactions. For an "even fixed point", the matrix $T^*_{\alpha\beta}$ breaks up into an even–even and an odd–odd part as can be seen by differentiating (1.11) with respect to a K_α:

$$[\partial G/\partial K_\alpha + \partial \mathcal{H}'(s')/\partial K_\alpha]\, e^{\mathcal{H}'(s')} = \sum_{\{s\}} P(s', s)\, e^{\mathcal{H}(s)} [\partial \mathcal{H}(s)/\partial K_\alpha] \tag{2.2}$$

Then, using that

$$\partial \mathcal{H}(s)/\partial K_\alpha = \sum_{a \in \alpha} s_a$$

and

$$\partial \mathcal{H}'(s')/\partial K_\alpha = \sum_{\alpha'} [\partial K'_{\alpha'}/\partial K_\alpha] \left[\sum_{a' \in \alpha'} s'_{a'} \right] \tag{2.3}$$

one may write (2.2) in the form

$$\partial G/\partial K_\alpha + \sum_{\alpha'} \left[\sum_{a' \in \alpha'} s'_{a'} \right] [\partial K'_{\alpha'}/\partial K_\alpha] = \left\langle \sum_{a \in \alpha} s_a \right\rangle_{s'} \tag{2.4}$$

where the average with the subscripts $\langle\ \rangle_{s'}$, to remind one of the dependence on $\{s'\}$, is defined as

$$\langle A \rangle_{s'} = \sum_{\{s\}} P(s', s)\, e^{\mathcal{H}(s)}\, A(s) / \sum_{\{s\}} P(s', s)\, e^{\mathcal{H}(s)} \tag{2.5}$$

From (2.5) it follows that for a Hamiltonian $\mathcal{H}(s)$ with only even interactions, $\langle A \rangle_{s'}$ and A will have the same symmetry. So in (2.4) for α even only even α' appear, and for odd α only odd α' will appear. Thus $T^*_{\alpha\beta}$ has even and odd eigenvalues and related even and odd scaling fields.

The distinction between scaling fields in odd and even symmetry character can be extended to more complicated symmetries; this will be discussed in connection with the symmetric eight-vertex model in Section VI.

In this Section we discuss:

A. the connection between the eigenvalues of $T^*_{\alpha\beta}$ and critical exponents;
B. the determination of the critical temperature;
C. the calculation of the free energy from (2.1);
D. the spontaneous magnetization.

A. Eigenvalues and critical exponents

We shall start off with a short reminder of the nomenclature of a few critical exponents for magnetic systems. For the sake of convenience we shall restrict ourselves to ferromagnetic systems; extensions to antiferromagnetic systems can be found e.g. in Stanley (1971).

We define the dimensionless distance τ from the critical temperature T_c by

$$\tau = (T - T_c)/T_c \tag{2.6}$$

The singular behaviour of the specific heat, spontaneous magnetization, magnetic susceptibility and the response to an external magnetic field at T_c are respectively characterized by the exponents α, β, γ and δ: $C_H \simeq \tau^{-\alpha}$, $m \simeq (-\tau)^{\beta}$, $\chi_H \simeq \tau^{-\gamma}$ and $H = |m|^{\delta}$. If $\tau < 0$ the critical exponents usually are primed. In Table II.1 we give the numerical values of the critical exponents for the Ising-like systems defined by the Hamiltonian (1.7) on a quadratic and a triangular lattice (1.8) and that of the symmetric eight-vertex model defined by (1.9).

TABLE II.1. Critical exponents for several Ising-like systems. The models have been defined in the text. Those marked by * and † are conjectures by Suzuki (1974) and Barber and Baxter (1973) respectively. $s(\lambda) = \pi/[\pi - \arccos(\tanh 2\lambda)]$; λ has been defined in (1.9).

Model	$\alpha = \alpha'$	β	$\gamma = \gamma'$	δ
2-d Ising	0(log)	1/8	7/4	15
2-d triplet interation	2/3	1/12*	7/6*	15*
symmetric 8-vertex	$2 - s(\lambda)$	$s(\lambda)/16$†	$7s(\lambda)/8$†	15†

The connection between the eigenvalues λ_i of the matrix $T^*_{\alpha\beta}$ and the critical exponents is most easily established by asking whether a singular behaviour in powers of the u_i for the free energy is compatible with (2.1). To begin with let us put $u_2 = u_3 = \ldots = 0$ and assume a behaviour like $A|u_1|^{a_1}$ for $f(u_1, 0, 0, \ldots)$. Inserting this into (2.1) and equating the powers of $u_1^{a_1}$ (which by the regularity assumption of $g(u_1, 0, 0, \ldots)$ do not appear in g) leads to the equality

$$A|u_1|^{a_1} = l^{-d}A[\lambda_1 u_1]^{a_1} \tag{2.7}$$

implying for the exponent a_1

$$a_1 = \mathrm{d}\log l/\log|\lambda_1|. \tag{2.8}$$

The conclusion on the one hand is that if a singular power in u_1 would occur, the exponent a_1 is given by λ_1 through (2.8). On the other hand we see from (2.8) that only the relevant scaling fields ($\lambda_i > 1$) qualify on physical grounds for a singular behaviour as the exponent should be positive to keep the free energy finite at $u_1 = 0$. The exponent a_1 is a physical quantity and thus λ_1 should vary as a power of l for different renormalization transformations with different l. Therefore, following Wegner (1972), we write for the (relevant) eigenvalues (which are assumed to be real and positive)

$$\lambda_i = l^{y_i} \tag{2.9}$$

such that a_1 and y_1 are related as

$$a_1 = d/y_1. \tag{2.10}$$

We note that the amplitude A drops out of (2.7). Since A could be zero, we may not invert the argument. A relevant eigenvalue does not necessarily lead to a singular behaviour. This point is investigated further under C.

Next we look into the dependence on two scaling fields u_1 and u_2 and determine the singular powers $|u_1|^{a_1}|u_2|^{a_2}$ which are allowed by (2.1). Comparing exponents left and right yields the relation

$$a_1 y_1 + a_2 y_2 = d. \tag{2.11}$$

For positive a_i at least one of the y_i has to be positive. A combination of a relevant scaling field ($y_1 > 0$) and an irrelevant one ($y_2 < 0$) leads to a value of a_1 larger than d/y_1 and therefore is a less singular contribution in u_1 than the previously found power d/y_1. Such corrections to the most singular part will be discussed in C. For the remaining case of two relevant scaling fields a combination of the possible powers $u_1^{a_1} u_2^{a_2}$ forms with (2.11) the functions

$$|u_1|^{d/y_1} f_1(u_2/|u_1|^{y_2/y_1}) \tag{2.12}$$

or

$$|u_2|^{d/y_2} f_2(u_1/|u_2|^{y_1/y_2}). \tag{2.13}$$

These are solutions of the scaling relation

$$f_{\text{sing}}(l^{y_1} u_1, l^{y_2} u_2) = l^d f_{\text{sing}}(u_1, u_2). \tag{2.14}$$

Except for the fact that here l is a discrete variable, (2.14) is similar to Widom's static scaling hypothesis (Widom, 1965a, b), which is usually written in the form

$$f_{\text{sing}}(x^{a_\varepsilon} \tau, x^{a_h} h) = x f_{\text{sing}}(\tau, h) \tag{2.15}$$

where h is a dimensionless magnetic field.

In order to make the connection with the usual scaling theory of critical phenomena we have to show that in the appropriate cases a fixed point has two relevant variables u_1 and u_2, one of which, e.g. u_1, is temperature-like and the other, u_2, field-like.

As has already been mentioned (if the Hamiltonian contains only even interactions), a "normal" fixed point will lie in the subspace of even interactions. A change of temperature is a change of magnitude of all interaction constants K_α. Thus a variation of the temperature in the fixed point keeps the Hamiltonian in the even subspace and the temperature will couple to the even scaling fields. Similarly a change of the magnetic field couples to the odd scaling fields. We leave the confirmation of this picture to Section V, where several specific cases are treated, and give here only the relation of the thermodynamic critical exponents and the two exponents y_T (even) and y_H (odd) of the relevant variables

$$\begin{aligned} \alpha &= 2 - d/y_T \\ \beta &= (d - y_H)/y_T, \\ \gamma &= (2y_H - d)/y_T, \\ \delta &= y_H/(d - y_H), \end{aligned} \tag{2.16}$$

from which immediately follow the well-known scaling laws

$$\alpha + 2\beta + \gamma = 2 \quad \text{and} \quad \alpha + \beta(\delta + 1) = 2. \tag{2.17}$$

We note here again that the relation (2.14) does not determine the functions f_1 or f_2. A determination of f_1 and f_2 follows in C.

As noted before, a difference between (2.14) and (2.15) is that x is a continuous parameter whereas l has a fixed value for a renormalization transformation.

For more than two relevant variables (which happens in special fixed points) one has similarly to (2.14) for the singular part the scaling relation:

$$f_{\text{sing}}(l^{y_1}u_1, l^{y_2}u_2, \ldots, l^{y_n}u_n) = l^d f_{\text{sing}}(u_1, u_2, \ldots, u_n). \tag{2.18}$$

Up till now we have only been discussing the behaviour in the fixed point and we must extend the conclusion to the domain of attraction, which is by (2.1) thermodynamically equivalent to the fixed point. However, having observed that the (possible) singularities occur for small values of the relevant scaling fields, we may already conclude that the critical systems must be found in the domain of attraction as this is the collection of systems with all relevant scaling fields $u_1 = u_2 = \ldots = u_n = 0$.

B. Determination of the critical temperature

The domain of attraction or the surface of criticality is determined by setting the relevant scaling fields equal to zero. Restricting ourselves to the even subspace and assuming that there is only one relevant eigenvalue λ_T with corresponding scaling field u_T, the equation for the surface of criticality becomes with (1.32)

$$0 = u_T = \sum_\alpha \varphi_\alpha^T (K_\alpha - K_\alpha^*) + \frac{1}{2!} \sum_{\alpha\beta} \varphi_{\alpha\beta}^T (K_\alpha - K_\alpha^*)(K_\beta - K_\beta^*) + \cdots \tag{2.19}$$

For a given set of interaction parameters $J_\alpha = k_B T K_\alpha$ equation (2.19) then determines the critical temperature $T_c(J)$. Writing $K_\alpha = J_\alpha / k_B T_c(J)$ and substituting in (2.19) the function $T_c(J)$ has to be such that (2.19) is fulfilled.

If one approximates the surface of criticality by a plane which is tangent to it in the fixed point, (2.19) reduces to the following equation for $T_c(J)$

$$k_B T_c(J) = \sum_\alpha \varphi_\alpha^T J_\alpha / \sum_\alpha \varphi_\alpha^T K_\alpha^*. \tag{2.20a}$$

This yields the critical temperature with the fixed point as reference point. Dalton and Wood (1969) have studied the variation of the critical temperature $T_c(J)$ due to the admixture of nearest-neighbour interactions taking the "pure" Ising system (1.7) as a reference point. In order to compare (2.20a) with their data we first notice that (2.20a) yields for the pure Ising system critical temperature T_c

$$k_B T_c = \varphi_{\text{n.n.}}^T J_{\text{n.n}} / \sum_\alpha \varphi_\alpha^T K_\alpha^*, \tag{2.20b}$$

where the subscript n.n. stands for nearest neighbour. Then using (2.20b) in (2.20a) one obtains the equation for $T_c(J)$ in the form of Dalton and Wood (1969):

$$T_c(J) = T_c \left\{ 1 + \sum_{\alpha \neq \text{n.n}} (\varphi_\alpha^T / \varphi_{\text{n.n}})(J_\alpha / J_{\text{n.n}}) \right\}. \tag{2.20c}$$

C. Calculation of the free energy†

From the previous section it follows that from (2.1) alone, the free energy cannot be determined in terms of $g(u_1, u_2, \ldots)$ since the singular part fulfils the homogeneous equation with $g = 0$. As in the theory of differential equations one needs a boundary condition to specify the amplitude of the solution of the inhomogeneous equation which has to be added to a particular solution of the inhomogeneous equation.

A point in the neighbourhood of the domain of attraction (see Fig. 2)

† Nauenberg and Nienhuis (1974b), Nightingale and 't Hooft (1975), Nelson and Fisher (1975), and van Leeuwen (1975a).

of a non-trivial fixed point first moves, upon repeated application of the renormalization transformation, along the domain of attraction towards the fixed point. If it originally was not exactly in the domain of attraction, it bends away from the fixed point, usually towards another fixed point with a higher dimensional domain of attraction. For the ferromagnetic fixed point in the even subspace a point outside the domain of attraction moves either to a region with very large interaction parameters (i.e. $T = 0$) or to a region with very small interaction parameters (i.e. $T = \infty$). Both limiting situations can be considered as fixed points. The behaviour of the free energy in the neighbourhood of these (trivial) fixed points determines the singular behaviour in the neighbourhood of the domain of attraction of the original fixed point. It is of course possible that there are several fixed points, each having its own domain of attraction, which leads to a more complicated picture.

Let us illustrate the general procedure by taking the following one-dimensional example. Consider the subspace of one relevant (even) scaling field u_1 setting all other $u_2 = u_3 = \ldots = 0$. Then (2.1) simplifies to

$$f(u) = g(u) + l^{-d} f(\lambda u) \tag{2.21}$$

(dropping the index 1 of u_1). Iterating (2.21) n times yields

$$f(u) = \sum_{j=0}^{n-1} l^{-jd} g(\lambda^j u) + l^{-nd} f(\lambda^n u) \tag{2.22}$$

Now we take as boundary condition that we may let $n \to \infty$ in (2.22) and drop the term $l^{-nd} f(\lambda^n u)$ in that limit:

$$\lim_{n \to \infty} l^{-nd} f(\lambda^n u) = 0, \tag{2.23}$$

Whether this is permitted or not must be verified in specific cases. The idea behind this boundary condition is that the free energy behaves sufficiently well for large values of the scaling fields ($T = 0$ or $T = \infty$), in the sense that it does not compensate the power l^{-nd}. Thus we take for f the expression

$$f(u) = \sum_{j=0}^{\infty} l^{-jd} g(\lambda^j u). \tag{2.24}$$

Without detailed calculation this expression already suggests that $u = 0$ is a possible singular point. For $u > 0$, $f(u)$ is determined by the values of g for $u > 0$, and f for $u < 0$ is determined by g for $u < 0$. If there is no symmetry relation between g for $u > 0$ and $u < 0$, two different functions f may result for $u > 0$ and $u < 0$ respectively, which join continuously at the value

$$f(0) = \sum_{j=0}^{\infty} l^{-jd} g(0) = g(0)/(1 - l^{-d}). \tag{2.25}$$

We shall define the scaling field u in such a way that $u > 0$ corresponds to temperatures below the critical temperature.

We shall now first construct the regular part $f_{\text{reg}}(u)$ of $f(u)$. Since $g(u)$ is a regular function it will, by definition, have a power series expansion

$$g(u) = \sum_{n=0}^{\infty} g_n u^n. \tag{2.26}$$

Inserting this into (2.24) and substituting for $f(u)$ yields

$$f_{\text{reg}}(u) = \sum_{n=0}^{\infty} f_n u^n \tag{2.27}$$

One finds the coefficients f_n as

$$f_n = g_n/(1 - l^{-d}\lambda^n). \tag{2.28}$$

We see that the case $\lambda^n = l^d$ must be discussed separately as will be done at the end of this section. f_{reg} is a particular solution of (2.21) characterized by the fact that it is regular at the origin. From (2.28) we see that the radius of convergence of f is larger by a factor λ than that of g.

Returning to the expression (2.24) we shall denote by n_0 the first power for which $\lambda^n > l^d$ and write

$$g(u) = \sum_{n=0}^{n_0-1} g_n u^n + g_{\text{rem}}(u) \tag{2.29}$$

(where the subscript "rem" stands for remainder). We note that for an irrelevant scaling field ($\lambda < 1$), λ^n will never exceed l^d, such that $n_0 = \infty$ and $g_{\text{rem}}(u)$ will disappear. Inserting (2.29) into (2.24) one observes that for the first n_0 powers the summation over j converges and using (2.28) one finds

$$f(u) = \sum_{n=0}^{n_0-1} f_n u^n + \sum_{j=0}^{\infty} l^{-jd} g_{\text{rem}}(\lambda^j u) \tag{2.30}$$

For the remainder we write

$$\sum_{j=0}^{\infty} l^{-jd} g_{\text{rem}}(\lambda^j u) = \sum_{j=-\infty}^{\infty} l^{-jd} g_{\text{rem}}(\lambda^j u) - \sum_{j=-\infty}^{-1} l^{-jd} g_{\text{rem}}(\lambda^j u) \tag{2.31}$$

The part that is added and subtracted can be evaluated by a power series expansion of $g_{\text{rem}}(u)$

$$\sum_{j=-\infty}^{-1} l^{-jd} g_{\text{rem}}(\lambda^j u) = \sum_{m=n_0}^{\infty} f_m u^m \tag{2.32}$$

Collecting (2.30), (2.31) and (2.32) together one arrives at

$$f(u) = f_{\text{reg}}(u) + f_{\text{sing}}(u) \tag{2.34}$$

with $f_{\text{sing}}(u)$ given as

$$f_{\text{sing}}(u) = \sum_{j=-\infty}^{\infty} l^{-jd} g_{\text{rem}}(\lambda^j u) \tag{2.34}$$

Of course $f_{\text{sing}}(u)$ fulfils the condition of homogeneity

$$f_{\text{sing}}(\lambda u) = l^d f_{\text{sing}}(u) \tag{2.35}$$

since $f_{\text{reg}}(u)$ is a solution of (2.21). Equation (2.34) is an explicit solution of (2.35). It should be noted that only the remainder of g appears in (2.34), otherwise the summation could not be extended to $j = -\infty$ and that in turn is necessary to fulfil (2.35). On the upper side $j = \infty$ the convergence is determined by the behaviour of $g(u)$ for large u, since the subtracted terms do not influence the convergence. This again has to be checked in every specific case. If the convergence at the upperside $j = \infty$ would become questionable it should be taken as a signal that the boundary condition in (2.23) has to be reconsidered.

Having obtained an explicit expression for f_{sing} we are in a position to discuss the question whether f_{sing} indeed behaves as $A|u|^{d/y}$ and to determine the value of the amplitude A. We define an amplitude $A(u)$ as

$$A(u) = |u|^{-d/y} \sum_{j=-\infty}^{\infty} l^{-jd} g_{\text{rem}}(\lambda^j u). \tag{2.36}$$

$A(u)$ then automatically satisfies the relation $A(\lambda u) = A(u)$. Thus $A(u)$ takes on equal values in all arguments $\ldots \lambda^{-2}u, \lambda^{-1}u, u, \lambda u, \lambda^2 u, \ldots$ of which sequence $u = 0$ (the fixed point) is a limit point. This, however, is not sufficient to guarantee that $A(u)$ is a constant (depending only on the sign of u), since (2.36) allows for a periodic function in $\log|u|$ with period $\log\lambda$. Thus we write

$$\begin{aligned} A(u) &= \sum_{n=-\infty}^{\infty} A_n^{\pm} \exp\left[2\pi i n \log|u|/\log\lambda\right] \\ &= \sum_{n=-\infty}^{\infty} A_n^{\pm} |u|^{2\pi i n/\log\lambda}. \end{aligned} \tag{2.37}$$

The signs $\pm$ are introduced to distinguish the functions $A^{\pm}(u)$ for $u \gtrless 0$.
The fourier coefficients are obtained from (2.36) as

$$A_n^{\pm} = \frac{1}{\log\lambda} \int_0^{\log\lambda} \mathrm{d}\log u \left\{ |u|^{-d/y} \sum_{j=-\infty}^{\infty} l^{-jd} g_{\text{rem}}(\lambda^j u) \right\} |u|^{-2\pi i n/\log\lambda}. \tag{2.38}$$

Changing to the variable $t = \lambda^j |u|$ in the jth term, the summation over j leads to a sum over intervals in t which spans the range $[0, \infty]$

$$A_n^{\pm} = \frac{1}{\log \lambda} \int_0^{\infty} \frac{\mathrm{d}t}{t^{(1+d/y)}} g_{\mathrm{rem}}(\pm t) t^{-2\pi i n/\log \lambda}. \tag{2.39}$$

The result (2.37) with $A_n^{\pm}$ given by (2.39) is somewhat disconcerting since it does not follow from other renormalization procedures. However, it is unlikely that the oscillatory terms ($n \neq 0$) will actually exist, because they imply that the free energy has an oscillation in $\log |u|$ of period $\log \lambda$, superimposed on the power $|u|^{d/y}$. But λ contains the scaling length l which can be changed by choosing a different renormalization transformation. In fact, if two renormalization transformations exist with asymptotic, i.e. for $u \to 0$, equivalent scaling fields and incommensurate $\log l_1$ and $\log l_2$, such oscillating terms cannot exist.† But if we stick to one particular renormalization transformation, as we shall do in this paper, we can only hope that the $A_n^{\pm}$ for $n \neq 0$ are found to be zero from (2.39). Actual calculations so far have always led to no noticeable $A_n^{\pm}$ for $n \neq 0$.

So we assume that only $A_0^{\pm}$ exists with the result that

$$A(u) = A_0^{\pm} = \frac{1}{\log \lambda} \int_0^{\infty} \frac{\mathrm{d}t}{t(1+d/y)} g_{\mathrm{rem}}(\pm t) = A^{\pm}. \tag{2.40}$$

Thus the resulting $f_{\mathrm{sing}}(u)$ reads, with (2.34), (2.36), and (2.40),

$$f_{\mathrm{sing}}(u) = A^{\pm} |u|^{d/y}, \qquad u \gtrless 0. \tag{2.41}$$

The general situation, with more than one scaling field present, can be treated quite similarly. With the same boundary condition

$$\lim_{n \to \infty} l^{-nd} f(\lambda_1^n u_1, \lambda_2^n u_2, \ldots) = 0 \tag{2.42}$$

the solution of (2.1) is given by

$$f(u_1, u_2, \ldots) = \sum_{j=0}^{\infty} l^{-jd} g(\lambda_1^j u_1, \lambda_2^j u_2, \ldots). \tag{2.43}$$

Again g is split into two parts: one part containing the power combinations for which (2.43) converges and a remainder. The first part leads to a piece of the regular part of the free energy $f_{\mathrm{reg}}(u_1, u_2, \ldots)$ which is constructed from the power series of $g(u_1, u_2, \ldots)$

$$g(u_1, u_2, \ldots) = \sum_{n_1, n_2, \ldots = 0}^{\infty} g_{n_1, n_2} u_1^{n_1} u_2^{n_2} \ldots \tag{2.44}$$

as

$$f_{\mathrm{reg}}(u_1, u_2, \ldots) = \sum_{n_1, n_2, \ldots = 0}^{\infty} f_{n_1, n_2, \ldots} u_1^{n_1} u_2^{n_2} \ldots \tag{2.45}$$

† This argument is due to D. R. Nelson (private communication). The possible existence of oscillatory terms was noted by Nauenberg (1975b).

with $f_{n_1, n_2, \ldots}$ given by

$$f_{n_1, n_2, \ldots} = g_{n_1, n_2, \ldots}/(1 - l^{-d}\lambda_1^{n_1}\lambda_2^{n_2} \ldots) \tag{2.46}$$

In the second part of the summation over $j = -\infty$ to -1 is added and subtracted and evaluating this sum through power series expansion one recovers the second piece of f_{reg}. So again one finds f as

$$f(u_1, u_2, \ldots) = f_{\text{reg}}(u_1, u_2, \ldots) + f_{\text{sing}}(u_1, u_2, \ldots) \tag{2.47}$$

with f_{sing} given by

$$f_{\text{sing}}(u_1, u_2, \ldots) = \sum_{j=-\infty}^{\infty} l^{-jd} g_{\text{rem}}(\lambda_1^j u_1, \lambda_2^j u_2, \ldots) \tag{2.48}$$

f_{sing} fulfils the homogeneity property.

An interesting aspect of more than one scaling field is the interplay of the relevant variables and the rôle of the irrelevant variables. If we take u_1 as a relevant variable and write $u_2 = v_2|u_1|^{y_2/y_1}$, $u_3 = v_3|u_1|^{y_3/y_1}$ etc., then

$$A_1(v_2, v_3, \ldots; u_1) = |u_1|^{-d/y_1} \sum_{j=-\infty}^{\infty} l^{-jd} g_{\text{rem}}(\lambda_1^j u_1, \lambda_2^j v_2 |u_1|^{y_2/y_1}, \ldots) \tag{2.49}$$

has the property that $A_1(v_2, v_3, \ldots; \lambda_1 u_1) = A_1(v_2, v_3, \ldots; u_1)$, as before implying that A_1 is periodic in $\log|u_1|$ with period $\log \lambda_1$. We shall again assume that the oscillatory terms vanish and that A_1 is completely determined by its zeroth fourier component, which reads in analogy with (2.40)

$$A_1^{\pm}(v_2, v_3, \ldots) = \frac{1}{\log \lambda_1} \int_0^{\infty} \frac{\mathrm{d}t}{t^{(1+d/y)}} g_{\text{rem}}(\pm t, v_2 t^{y_2/y_1}, v_3 t^{y_3/y_1}, \ldots). \tag{2.50}$$

Thus we obtain for f_{sing} the expression

$$f_{\text{sing}}(u_1, u_2, u_3, \ldots) = |u_1|^{d/y_1} A_1^{\pm}(u_2/|u_1|^{y_2/y_1}, u_3/|u_1|^{y_3/y_1}, \ldots). \tag{2.51}$$

This is a generalization of (2.12). We now point out a difference between the relevant and irrelevant variables. Suppose that u_2 is another relevant variable and u_3, etc. irrelevant variables. Then $y_3 < 0, \ldots$ and for small u_1 the second, third etc. argument become small. To leading order we have

$$f_{\text{sing}}^{\text{lead}}(u_1, u_2, u_3, \ldots) = |u_1|^{d/y_1} A_1^{\pm}(u_2/|u_1|^{y_2/y_1}, 0, 0 \ldots) \tag{2.52}$$

showing that the most singular part is independent of the irrelevant variables. So around the domain of attraction we have to leading order a universal singular behaviour in the relevant variables. Eqn (2.52) shows that (2.12) is not only restricted for the fixed point but holds for the whole domain of attraction, which is therefore correctly identified with the *surface of criticality*.

For the next order we obtain (due to the u_3-dependence)

$$f_{\text{sing}}^{\text{next}}(u_1, u_2, u_3, \ldots) = u_3 |u_1|^{(d-y_3)/y_1} \left[\frac{\partial A_1^{\pm}}{\partial v_3} (u_2/|u_1|^{y_2/y_1}, v_3, \ldots) \right]_{v_3=0} \tag{2.53}$$

and similar expressions for the corrections to scaling (Wegner, 1972) due to the other variables. The largest y_i (<0) yields the smallest difference in power of $|u_1|$ with respect to the leading power.

For the relevant variable u_2 we have to keep the whole dependence of A_1 on v_2, since $u_2/|u_1|^{y_2/y_1}$ may be small or large depending on the values of u_2 and u_1. A has been given a subscript 1 simply because u_1 has been singled out as a relevant variable. Choosing u_2 as a relevant variable would have led to different expressions (of the second type of (2.13)) where the rôles of u_1 and u_2 would have been interchanged.

Finally we have to consider the case, as mentioned before, that a power λ^n is equal to l^d, or more generally that a combination $\lambda_1^{n_1}\lambda_2^{n_2}\ldots = l^d$. Now one can follow two different procdures: either treat the "dangerous" power combination separately from the start or take the limit in the final formulae as obtained by treating (one of) the eigenvalue(s) as a continuous parameter. In either case the result is the appearance of a logarithmic term. As the derivation closely parallels the one already given, we shall present here only the result.

Define $f_{\text{reg}}(u_1, u_2, \ldots)$ as before with the exception that for the coefficient $f_{n_1, n_2, \ldots}$ of the "dangerous" power for which $\lambda_1^{n_1}\lambda_2^{n_2}\ldots = l^d$, not the relation (2.46) is used but instead

$$f_{n_1, n_2, \ldots} = \tfrac{1}{2} g_{n_1, n_2, \ldots} \tag{2.54}$$

Now $g_{\text{rem}}(u_1, u_2, \ldots)$ is defined as $g(u_1, u_2, \ldots)$ minus the "dangerous" power and the summable powers. The amplitude $A_1^{\pm}(v_1, v_2, \ldots)$ is defined as

$$A_1^{\pm}(v_2, v_3, \ldots) = \frac{1}{\log \lambda_1} \left[\int_0^1 \mathrm{d}t \, \frac{g_{\text{rem}}(\pm t, v_2 t^{y_2/y_1}, \ldots)}{t^{(1+d/y_1)}} \right.$$
$$\left. + \int_1^{\infty} \mathrm{d}t \, \frac{g_{\text{rem}}(\pm t, v_2 t^{y_2/y_1}, \ldots) + g_{n_1, n_2 \ldots}(\pm t)^{n_1}(v_2 t^{y_2/y_1})^{n_2} \ldots}{t^{(1+d/y_1)}} \right]. \tag{2.55}$$

Then the singular part of the free energy reads:

$$f_{\text{sing}}(u_1, u_2, \ldots) = |u_1|^{d/y_1} A_1^{\pm}(u_2/|u_1|^{y_2/y_1}, \ldots)$$
$$- \frac{g_{n_1, n_2, \ldots}}{\log \lambda_1} u_1^{n_1} u_2^{n_2} \ldots \log |u_1| \tag{2.56}$$

Here of course $n_1, n_2 \ldots$ is the combination for which $\lambda_1^{n_1}\lambda_2^{n_2}\ldots = l^d$. For the case of a single relevant variable, $u_2 = u_3 = \ldots = 0$, the first term in (2.56) is a regular power of u_1 since then d/y is an integer and the singular part of

the free energy is a pure logarithmic term. As with most logarithmic terms, its coefficient is simply given by the coefficient of the "dangerous" power in $g(u_1, u_2, \ldots)$. Whereas in general the coefficient of the singular power $|u_1|^{d/y_1}$ is different for $u_1 > 0$ and $u_1 < 0$ respectively (i.e. below and above T_c), the coefficient of the logarithmic term is equal, as was already noted by Widom (1965a, b).

D. The spontaneous magnetization

So far only critical singularities have been discussed. However, in a ferromagnetic phase transition the critical point is also the onset of the appearance of a spontaneous magnetization for lower temperatures. In terms of the free energy the problem is to understand from (1.16) why, for zero magnetic field H, the derivative $\partial f/\partial H$ (magnetization) vanishes for $T > T_c$ and has a non-zero value for $T < T_c$. In order to discuss this phenomenon we have to restrict ourselves to a ferromagnetic phase transition. How a ferromagnetic phase transition manifests itself in the context of the renormalization theory has yet to be explained. To simplify the discussion we restrict ourselves to a renormalization transformation where the weights $P(s', s)$ are either 1 or 0, such that a well-defined class of allowed site spin configurations belongs to every cell spin configuration.

We will sketch here, without proof, the picture that emerges from the example to be treated in Section V. Consider a fixed point in the even subspace and a neighbouring point in the direction of the relevant (even) eigenvector. We take the direction of the eigenvector such that it points to systems with larger K_α (i.e. in the direction of lower temperatures). If the neighbouring point is repeatedly transformed the K_α will grow indefinitely to arbitrarily large values. On the other hand, a point close to the fixed point but in the direction opposite to the relevant eigenvector will decrease to vanishing values of K_α. In the latter case the point approaches the trivial fixed point $K = 0$ (∞ temperature). We shall argue that in the low-temperature limit also a new fixed point is approached, which has, however, the somewhat awkward position $K = \infty$ or at least a position with some of the $K_\alpha = \infty$.

The ∞-temperature properties of the matrix $\partial K'_{\alpha'}/\partial K_\alpha$ are easily obtained by the cumulant expansion given in the next Section. It suffices here to state that all the eigenvalues stay below l^d. The $T = 0$ limit is more difficult to discuss. The key idea is that for the transformation (1.11) in the limit $K_\alpha \to \infty$ *one* (or a finite number) of the configurations $\{s\}$ starts to contribute dominantly. In appendix B it will be shown that excitations with respect to this dominant configuration contribute only to order $\exp(-\Delta)$ where Δ is the energy of the excitation. The dominant state $\{\bar{s}\} = \{\bar{s}(s')\}$ does depend on the cell state $\{s'\}$. In fact it is the state which maximizes $P(s', s) \exp \mathcal{H}(s)$ as function of $\{s\}$.

For the triangular lattice transformation (1.18) it is possible to substantiate this idea by examples. We denote by $\{s_+\}$ the site spin configuration with all spins $s_i = 1$ and by $\{s_-\}$ the configuration with all $s_i = -1$. When all the $K_\alpha \geqslant 0$ (for even and odd α) the state $\{s_+\}$ maximizes $\mathscr{H}(s)$. For the cell spin configuration $\{s'_+\}$ the site spin configuration $\{s_+\}$ is allowed and thus

$$\{\bar{s}(s'_+)\} = \{s_+\}, \qquad K_\alpha \geqslant 0. \tag{2.57}$$

When $\{s'\}$ differs from $\{s'_+\}$ in one cell i' with $s'_{i'} = -1$, we must find $\{\bar{s}(s')\}$ between the site spin configurations with at least two site spins down in the cell i'. Without the relative strengths of the K_α known, one does not know whether $\mathscr{H}(s)$ possibly increases by turning some more spins, but when the magnetic field is sufficiently strong one should keep the number of down spins to a minimum and the 3 states with only two spins down maximize $\mathscr{H}(s)$ for that $\{s'\}$. For the case where all even $K_\alpha \geqslant 0$ and odd $K_\alpha \leqslant 0$ one has

$$\{\bar{s}(s'_-)\} = \{s_-\} \begin{cases} K_\alpha \geqslant 0 & \alpha \text{ even} \\ K_\alpha \leqslant 0 & \alpha \text{ odd} \end{cases} \tag{2.58}$$

and similar considerations for $\{s'\}$ differing from $\{s'_-\}$ in a few cells.

Neglecting the excitations and using that $P(s', \bar{s}(s')) = 1$, the transformation reduces for low temperatures to

$$G + \mathscr{H}'(s') = \mathscr{H}(\bar{s}(s')) + \log n(s') \tag{2.59}$$

where $n(s')$ is the degeneracy of the maximizing state $\bar{s}(s')$. For sufficiently low temperature the $\log n(s')$ term becomes negligible with respect to the large energy term $\mathscr{H}$. Under the circumstances given in (2.57), viz. (2.58), $\mathscr{H}'(s')$ will also be maximal for $\{s'_+\}$ viz. $\{s'_-\}$ showing that these absolute maximum states $\{s_\pm\}$ (ground states) are invariant under the transformation. The subspace with the odd $K_\alpha = 0$ separates the two domains where $\{s_+\}$ and $\{s_-\}$ are the groundstates. In the even subspace $\mathscr{H}(s_+)$ and $\mathscr{H}(s_-)$ are equal.

Now we apply the low-temperature limit also to (2.4) in the same sense as (2.59), by only taking the dominant state(s) into account in the average $\langle \rangle'_{s'}$. Then (2.4) becomes for $T \to 0$

$$\partial G/\partial K_\alpha + \sum_{\alpha'} \left(\sum_{a' \in \alpha} s'_{a'} \right) (\partial K'_{\alpha'}/\partial K_\alpha) = [n(s')]^{-1} \sum_{\deg} \sum_{a \in \alpha} \bar{s}_a(s') \tag{2.60}$$

where $\Sigma_{\deg}$ is the sum over the degenerate maximizing states $\bar{s}(s')$. We consider (2.60) for $\{s'_\pm\}$ and write

$$\sum_{a \in \alpha} (s_\pm)_a = \chi_\alpha^\pm N. \tag{2.61}$$

Here $\chi_\alpha^\pm$ is the number of interactions of type α per site in state $\{s_\pm\}$. One has e.g.

$$\chi_\alpha^\pm = \pm 1 \quad \alpha = \text{single site}$$

$$\chi_\alpha^\pm = z/2 \quad \alpha = \text{nearest neighbours, } z = \text{coordination number.}$$

Then (2.60) becomes

$$\partial G/\partial K_\alpha + l^{-d}\sum_{\alpha'} \chi_{\alpha'}^\pm(\partial K_{\alpha'}/\partial K_\alpha) = \chi_\alpha^\pm \tag{2.62}$$

provided that (2.57) and (2.58) apply, which is still the case in the even subspace $K_\alpha \geqslant 0$ ($\bar{s}(s'_\pm)$ are non degenerate). Subtracting the $-$ from the $+$ equation yields

$$\sum_{\alpha'} (\chi_{\alpha'}^+ - \chi_{\alpha'}^-)(\partial K_{\alpha'}/\partial K_\alpha) = l^d(\chi_\alpha^+ - \chi_\alpha^-) \tag{2.63}$$

showing that $\chi_\alpha^+ - \chi_\alpha^-$ is an eigenvector with eigenvalue l^d. The eigenvector has components only in the odd directions, since $\chi_\alpha^+ = \chi_\alpha^-$ for the even α.

Equation (2.63) constitutes an important result. Going from a fixed point K^* along an eigendirection in the even subspace to a low-temperature regime where $K_\alpha \geqslant 0$, (2.63) applies and one moves towards a $T = 0$ fixed point having an eigenvalue l^d. The eigenvalue clearly arises out of competition of the two ferromagnetic states $\{s_+\}$ and $\{s_-\}$. A fixed point with these properties will be associated with a ferromagnetic phase transition. As we shall argue in the discussion, the transformation (1.18) is implicitly designed to focus attention on the ferromagnetic phase transition and so are most other proposed renormalization transformations.

With this preparation we now return to the problem of the spontaneous magnetization. Let K be a point in the even subspace and K_α an odd interaction constant. Differentiation of (1.16) with respect to K_α yields

$$\frac{\partial f(K)}{\delta K_a} = l^{-d}\sum_{\alpha'} \frac{\partial f(K')}{\partial K'_{\alpha'}} \frac{\partial K'_{\alpha'}}{\partial K_\alpha} \qquad (\alpha \text{ odd}) \tag{2.64}$$

where we use again the fact that $\partial g/\partial K_\alpha$ vanishes for α odd. Iterating (2.64) n times leads to

$$\frac{\partial f(K)}{\partial K_\alpha} = l^{-nd}\sum_{\alpha_n} \cdots \sum_{\alpha_n} \frac{\partial f(K^{(n)})}{\partial K_{\alpha_n}^{(n)}} \cdot \frac{\partial K_{\alpha_n}^{(n)}}{\partial K_{\alpha_{n-1}}^{(n-1)}} \cdots \frac{\partial K'_{\alpha'}}{\partial K_\alpha} \quad (\alpha \text{ odd}). \tag{2.65}$$

The derivative $\partial K_{\alpha_{j+1}}^{(j+1)}/\partial K_{\alpha_j}^{(j)}$ must be evaluated at the point $K^{(j)}$, the jth times transform of the initial point K.

Now first take K in the neighbourhood of the fixed point in the direction of (or opposite to) the relevant (even) eigenvector. As has been argued the

matrix $\partial K_{\alpha_n}^{(n)}/\partial K_{\alpha_{n-1}}^{(n-1)}$ will in both cases approach a limiting value. For $K^{(n)} \to 0$ the eigenvalues are smaller than l^d and the factor l^{-nd} in front of the expression (2.65) will give a zero limit for $n \to \infty$ as the derivatives $\partial f/\partial K_\alpha$ stay finite. For $K^{(n)} \to \infty$ the expression (2.65) yields a finite limit, since the matrices start to develop an eigenvalue l^d as seen from (2.63) for α odd. The limiting derivative $\partial f/\partial K_\alpha$ can also be found by observing that $f(K)$ is determined by the ground states $\{s_\perp\}$ as

$$f(K) = \sum_\alpha K_\alpha \chi_\alpha^\pm + N^{-1} \log 2. \tag{2.66}$$

An infinitesimal perturbation δK_α (α odd) removes the degeneracy (=2 for a ferromagnet: all up or down) and adds an amount

$$\delta f = \chi_\alpha^+ |\delta K_\alpha|.$$

So we conclude that for $K \to \infty$

$$\partial f(K)/\partial K_\alpha = \pm \chi_\alpha^+ = \chi_\alpha^\pm \qquad (\alpha \text{ odd}) \tag{2.67}$$

and we obtain for (2.59)

$$\frac{\partial f(K)}{\partial K_\alpha} = \pm \sum_{\alpha'} \chi_{\alpha'}^+ \lim_{n\to\infty} \left[\prod_{j=1}^{n} l^{-d} \partial K^{(j)}/\partial K^{(j-1)} \right]_{\alpha'\alpha} \tag{2.68}$$

Since l^d becomes an eigenvalue for $j \to \infty$ the infinite product will have a limiting value.

The result (2.68) is not limited to a specific path from the fixed point to $K = \infty$ but does equally well apply to points on the same temperature side of the critical surface. When only one even relevant eigenvalue is present points on the low-temperature side of the critical surface move towards the path along the relevant eigenvector ("gully bottom" in Wilson's (1971a, b) terminology). Similarly we find no spontaneous magnetization on the high-temperature side of the critical surface.

We point out that the expression (2.68) relates the derivatives to those of (2.67) at a more attractive fixed point. In this case the derivatives at $K = \infty$ could be determined directly, but in general one should also use the renormalization theory to determine the derivatives in terms of the derivatives of even more attractive fixed points, etc.

III. Correlation Functions

So far only the thermodynamic implications of the renormalization transformation (1.11) have been discussed. In this section we shall analyse the consequences for the correlation functions. The treatment given here differs from the usual discussion of correlation functions in renormalization theory.

because the arbitrariness of the weight function $P(s', s)$ does not permit us to follow the usual arguments, as we shall show below.

The standard correlation function $g(\mathbf{r})$ is defined as

$$g(\mathbf{r}_i - \mathbf{r}_j) = \langle s_i s_j \rangle \tag{3.1}$$

where $\mathbf{r}_i - \mathbf{r}_j$ is the relative position vector between sites i and j, which is measured in units of the site lattice spacing. For simplicity we restrict ourselves to even hamiltonians such that $g(\mathbf{r})$ decays to zero for large separations. We intend to establish a connection between $g(\mathbf{r})$ and the corresponding quantity $g'(\mathbf{r}')$ for the cell system. $g'(\mathbf{r}')$ is defined as

$$g'(\mathbf{r}'_{i'} - \mathbf{r}'_{j'}) = \langle s'_{i'} s'_{j'} \rangle' \tag{3.2}$$

where $\mathbf{r}'_{i'} - \mathbf{r}'_{j'}$ gives the relative position between the cells i' and j' measured in units of the cell lattice spacing. The primes on $g'(\mathbf{r}')$ and $\langle\ \rangle'$ are to remind us that the quantities refer to the renormalized hamiltonian $\mathscr{H}'(s')$.

It is not possible to derive directly a connection between the g and g' as was done for the free energies, unless $P(s', s)$ has special additional properties. To see this we form g' according to the prescription (3.2) and use the renormalization transformation (1.11)

$$\begin{aligned} g'(\mathbf{r}'_{i'} - \mathbf{r}'_{j'}) &= \sum_{\{s'\}} s'_{i'} s'_{j'} \exp\left[G + \mathscr{H}'(s')\right] / \sum_{\{s'\}} \exp\left[G + \mathscr{H}'(s')\right] = \\ &= \sum_{\{s'\}} \sum_{\{s\}} s'_{i'} s'_{j'} P(s', s) \exp \mathscr{H}(s) / \sum_{\{s\}} \exp \mathscr{H}(s) \end{aligned} \tag{3.3}$$

such that

$$g'(\mathbf{r}'_{i'} - \mathbf{r}'_{j'}) = \sum_{\{s'\}} s'_{i'} s'_{j'} \langle P(s', s) \rangle. \tag{3.4}$$

Without further specification of $P(s', s)$ the relation (3.4) is of little use. For a special class of $P(s', s)$, sometimes misleadingly called linear transformations, (3.4) is an interesting relation. We shall discuss this in part A. In B we return to the more general problem and give a simplified treatment of $g(\mathbf{r})$, which brings out the most essential features. The complete discussion is rather involved and is outlined in C. In the last part, D, we return to relation (3.4) and present an interesting connection between eigenvectors and critical exponents which is due to Kadanoff and Houghton (1975).

A. Linear weight-factors P(s', s)†

In practically all applications the weight-factor $P(s', s)$ is a product of weight-factors associated with the cells. Then $P(s', s)$ can be written as

$$P(s', s) = \prod_{i'} \tfrac{1}{2}(1 + s'_{i'} p(s^1_{i'}, s^2_{i'}, \ldots)). \tag{3.5}$$

† The relation of linear versus non-linear weight-factors is discussed by Bell and Wilson (1974).

Note that $P(s', s)$ fulfils condition (1.10b). In order that (1.10a) is fulfilled the function $|p|$ should not exceed 1 for any combination of its arguments. If $p(s_{i'}^1, \ldots)$ is a linear function of the spins of cell i', we call $P(s', s)$ a linear weight-factor (rather than calling the transformation based on $P(s', s)$ a linear linear transformation). Let us consider the simplest linear weight-factor

$$p(s_{i'}^1, s_{i'}^2, \ldots) = (s_{i'}^1 + s_{i'}^2 + \ldots)p \tag{3.6}$$

where p is as yet an arbitrary constant. Inserting (3.6) and (3.5) in (3.4) and first performing the summation over $\{s'\}$, one selects out of the product over the cells precisely the factors corresponding to cell i' and j' (all the other cell factors yield 1). So one finds

$$g'(\mathbf{r}'_{i'} - \mathbf{r}'_{j'}) = p^2 \langle (s_{i'}^1 + \ldots)(s_{j'}^2 + \ldots) \rangle'. \tag{3.7}$$

The interesting property of (3.6) is that $g'(\mathbf{r}')$ is related to $g(\mathbf{r})$ for different distances $\mathbf{r}$ formed by taking a site in cell i' and one in cell j'. Although in general complicated, (3.7) becomes simple when the distance between the cells i' and j' is large with respect to the difference in position inside the cells. Then

$$|\mathbf{r}_{i'}^s - \mathbf{r}_{j'}^t| \simeq r = lr' = l|\mathbf{r}'_{i'} - \mathbf{r}'_{j'}| \tag{3.8}$$

for all choices s and t out of cells i' and j'. Thus (3.7) becomes asymptotically

$$g'(r') \simeq p^2 l^{2d} g(lr') \tag{3.9}$$

(ignoring the possible angular dependence of $g(\mathbf{r})$).

This equation is very similar in structure to the general homogeneous equation for the free energy (2.35). Let us make the notation more explicit by considering the dependence on thermodynamic variables through a (relevant) scaling field u_T. Then (3.9) may be written as

$$g(u_T, r) = l^{-d+2-\eta} g(u'_T, r') = l^{-d+2-\eta} g(l^{y_T} u_T, l^{-1} r) \tag{3.10}$$

where we have put tentatively:

$$p^2 = l^{-d-2+\eta}. \tag{3.11}$$

Using the property (2.13) for generalized homogeneous functions we find for $g(u_T, r)$

$$g(u_T, r) = r^{-(d-2+\eta)} A(u_T r^{y_T}) \tag{3.12}$$

(one should realize that $1/r$ has the status of a scaling field for $r \to \infty$). Usually one writes the correlation function as a function of r/ξ, where ξ is the correlation length. (3.11) can be cast in that form by defining

$$\xi = u_T^{-1/y_T} \sim \varepsilon^{-\nu}. \tag{3.13}$$

This relation implies that the correlation length ξ depends asymptotically on the temperature difference τ with the exponent $\nu = 1/y_T$. The relation (3.11) is, however, curious, because it relates the physical exponent η to a seemingly arbitrary mathematical parameter p of the transformation (3.6). It would be better to reverse the argument and see relation (3.11) as a restriction on the choice of a meaningful p. Linear weights, more general than (3.6), lead to the same problem, to which we shall return in the discussion†.

For non-linear weight-factors e.g. (1.18) for the triangular lattice, the spin pair correlation function is, by (3.4), coupled to higher-order correlation functions (triplet–singlet and triplet–triplet correlation functions in the case of (1.18), as we shall show in *D*).

B. Simplified treatment of g(r)

The discussion of the correlation functions will be undertaken by considering them as derivatives of the free energy. Suppose we add a perturbation to the Hamiltonian $\mathscr{H}(s)$ of the type

$$\delta\mathscr{H}(s) = \sum_i \delta H_i s_i \tag{3.14}$$

which is a spatially dependent magnetic field H_i at site i. $\delta\mathscr{H}$ will lead to a change in the free energy δF

$$e^{F+\delta F} = \sum_{(s)} \exp\left[\mathscr{H}(s) + \delta\mathscr{H}(s)\right]. \tag{3.15}$$

Differentiating (3.15) with respect to δH_i and δH_j and setting $\delta H(s) = 0$ afterwards, yields the correlation function

$$G(\mathbf{r}_i - \mathbf{r}_j) = \delta^2 F/\delta H_i \delta H_j = \langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle. \tag{3.16}$$

Now we apply the renormalization transformation (1.11) to $\mathscr{H}(s) + \delta\mathscr{H}(s)$

$$\exp\left[G + \mathscr{H}'(s') + \delta C + \delta\mathscr{H}'(s')\right] = \sum_{(s)} P(s', s) \exp\left[\mathscr{H}(s) + \delta\mathscr{H}(s)\right] \tag{3.17}$$

and make the simplifying assumption that $\delta\mathscr{H}'(s')$ is given by

$$\delta\mathscr{H}'(s') = \sum_{i'} \delta H'_{i'} s'_{i'} \tag{3.18}$$

i.e. that only new (spatially dependent) magnetic fields $\delta H'_{i'}$ are generated for the cells by $\delta\mathscr{H}(s)$.

† The condition (1.10a) is violated by (3.11) since (1.10a) requires $|p| < l^{-d}$, while for $\eta > 2 - d$ (which holds for $d > 1$) (3.11) implies $|p| > l^{-d}$.

For the excess free energy δF we have corresponding to (1.13)†

$$\delta F = \delta C + \delta F'. \tag{3.19}$$

By (3.18) $\delta F'$ is a function of the $\delta H'_{i'}$ such that we have for (3.16)

$$G(\mathbf{r}_i - \mathbf{r}_j) = C(\mathbf{r}_i - \mathbf{r}_j) + \sum_{i'} \sum_{j'} G'(\mathbf{r}'_{i'} - \mathbf{r}'_{j'})(\delta H'_{i'}/\delta H_i)(\delta H'_{j}/\delta H_j) + \sum_{i'} (\delta F'/\delta H'_{i'})(\delta^2 H'_{i'}/\delta H_i \delta H_j) \tag{3.20}$$

with $C(\mathbf{r}_i - \mathbf{r}_j)$ defined by

$$C(\mathbf{r}_i - \mathbf{r}_j) = \delta^2 C/\delta H_i \delta H_j. \tag{3.21}$$

The derivative matrix

$$T_{i'i} = \delta H'_{i'}/\delta H_i \tag{3.22}$$

follows by differentiating (3.17) with respect to δH_i and use of (3.18)

$$\delta C/\delta H_i + \sum_{i'} s'_{i'} T_{i'i} = \langle s_i \rangle_{s'}, \tag{3.23}$$

where $\langle\ \rangle_{s'}$ is defined in (2.5).

Differentiating once more yields

$$C(\mathbf{r}_i - \mathbf{r}_j) + \sum_{i'} s'_{i'} \delta^2 H'_{i'}/\delta H_i \delta H_j = \langle s_i s_j \rangle_{s'} - \langle s_i \rangle_{s'} \langle s_j \rangle_{s'}. \tag{3.24}$$

For further simplification we may restrict ourselves to an even $\mathcal{H}(s)$. Then the averages $\langle\ \rangle_{s'}$ on the right-hand side of (3.23) and (3.24) are odd and even in s' respectively and thus $\delta C/\delta H_i$ and $\delta^2 H'_{i'}/\delta H_i \delta H_j$ vanish in that case. So (3.20) may be written as

$$G(\mathbf{r}_i - \mathbf{r}_j) = C(\mathbf{r}_i - \mathbf{r}_j) + \sum_{i'} \sum_{j'} G'(\mathbf{r}'_{i'} - \mathbf{r}'_{j'}) T_{i'i} T_{j'j}. \tag{3.25}$$

This is the desired transformation law for the correlation function which is the analogue of the free energy transformation (1.16). By choosing the symbols G and C we want to stress also the analogy with the Ornstein–Zernike relation, where the correlation function G is expressed in terms of the supposedly short-range direct correlation C. Here we shall also assume that $C(\mathbf{r})$ is short-ranged, which is a natural extension of the regularity assumption for $g(K)$ entering in the free energy equation. By summing (3.24) over j and using the derivative of (2.4) with respect to the magnetic field one finds the following relation between C and g.

† We use the same symbol G for the correlation function and the "empty set" contribution G to the free energy, which we hope will not lead to confusion. However, we have denoted the counterpart of the free energy G by C in (3.17) and (3.19) to avoid misunderstanding.

$$\sum_j C(\mathbf{r}_i - \mathbf{r}_j) = \frac{1}{N}\left[\left\langle \sum_i s_i \sum_j s_i \right\rangle_{s'} - \left\langle \sum_i s_i \right\rangle_{s'} \left\langle \sum_j s_j \right\rangle_{s'}\right] = \partial g^2/\partial H^2 \tag{3.26}$$

(ignoring in both (3.24) and (2.4) higher-order terms in s').

In fact one should run the argument backwards and base the regularity of g on the short-rangedness of $C(\mathbf{r})$. The right-hand side of (3.24) may be considered as the correlation function for a given cell configuration s' as constraint. Essentially one supposes that the *constrained* correlation function is short-range. The degrees of freedom left over in the partial average do not include the long-wavelength fluctuations which are responsible for the long-range correlations (and the thermodynamical singularities). These come about by the summation over the cell configurations, which amongst others describe the long-wavelength fluctuations. So the restricted averages on the right-hand side of (3.24) do not lead to long-range behaviour and associated thermodynamic singularities.

The analysis of (3.25) is, as for the Ornstein–Zernike equation, relatively simple, due to the translational invariance of $T_{i'i}$. We define the fourier transform $\tilde{T}(\mathbf{k})$ of $T_{i'i}$ as

$$\tilde{T}(\mathbf{k}) = \sum_i T_{i'i} \exp i\mathbf{k} \cdot (\mathbf{r}_{i'} - \mathbf{r}_i) \tag{3.27}$$

where both $\mathbf{r}_{i'}$ and $\mathbf{r}_i$, being the positions of cell i' and site i, are measured in units of the site lattice spacing. We note that $\tilde{T}(\mathbf{0})$ in the approximation (3.18) (or (3.23)) has the meaning of the magnetic eigenvalue λ_H, when worked out at the fixed point

$$\tilde{T}^*(\mathbf{0}) = \lambda_H = l^{y_H}. \tag{3.28}$$

Now multiplying (3.25) with $\exp i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)$ and summing over i and j yields

$$N\tilde{G}(\mathbf{k}) = N\tilde{C}(\mathbf{k}) + N'\tilde{T}(-\mathbf{k})\tilde{T}(\mathbf{k})\tilde{G}'(\mathbf{k}'). \tag{3.29}$$

The argument $\mathbf{k}'$ of $\tilde{G}'$ follows by rewriting the position vectors $\mathbf{r}_{i'}$ and $\mathbf{r}_{j'}$ in terms of the positions $\mathbf{r}'_{i'}$ and $\mathbf{r}'_{j'}$ which are measured in the cell lattice spacing. For some renormalization transformations we have a simple scale factor

$$\mathbf{r}_{i'} - \mathbf{r}_{j'} = l(\mathbf{r}'_{i'} - \mathbf{r}'_{j'}) \tag{3.30}$$

but there are cases (as in the triangular example) where the cell system is also rotated over an angle with respect to the site system. This complication is of little importance because in all examples we know of, one can do away with this rotation by considering higher powers of the transformation. So we restrict the discussion to case (3.30) which implies $\mathbf{k}' = \mathbf{k}l$. Thus (3.29) becomes

$$\tilde{G}(\mathbf{k}) = \tilde{C}(\mathbf{k}) + l^{-d}\tilde{T}(-\mathbf{k})\tilde{T}(\mathbf{k})\tilde{G}'(\mathbf{k}l). \tag{3.31}$$

The form (3.31) is almost the same in mathematical nature as the transformation law (1.16) for the free energy. The wave vector $\mathbf{k}$ plays the rôle of an additional scaling field with exponent $y = 1$. The difference from (1.16) is that the $\mathbf{k}$-dependent combination $l^{-d}\tilde{T}(-\mathbf{k})\tilde{T}(\mathbf{k})$ replaces the factor l^{-d}. This complication will be removed in a way that prepares for the general discussion of the next section. Define the regular function $\psi(\mathbf{k})$ through the recurrence relation

$$l^{y_H}\psi'(\mathbf{k}l) = \tilde{T}(\mathbf{k})\psi(\mathbf{k}) \tag{3.32}$$

with boundary condition $\psi^*(\mathbf{0}) = 1$. In order to explain (3.32) consider first this relation at the fixed point

$$l^{y_H}\psi^*(\mathbf{k}l) = \tilde{T}^*(\mathbf{k})\psi^*(\mathbf{k}). \tag{3.32a}$$

By power series expansion in $\mathbf{k}$ the $\psi^*(\mathbf{k})$ may be found for small $\mathbf{k}$ and then recursively for larger $\mathbf{k}$ by direct application of (3.32a) as in the case of scaling fields. Near the fixed point and for small $\mathbf{k}$ one may obtain $\psi(k)$ from (3.32) by simultaneous power series expansion in $\mathbf{k}$ and the scaling fields. Through application of (3.32) the functions $\psi(\mathbf{k})$ may be obtained further out.

Now multiply (3.31) by $\psi(-\mathbf{k})\psi(\mathbf{k})$ and with (3.32) one obtains

$$\tilde{\chi}(\mathbf{k}) = \tilde{\gamma}(\mathbf{k}) + l^{-d+2y_H}\tilde{\chi}'(\mathbf{k}l) \tag{3.33}$$

with the definitions

$$\begin{aligned} \tilde{\chi}(\mathbf{k}) &= \tilde{G}(\mathbf{k})\psi(\mathbf{k})\psi(-\mathbf{k}), \\ \tilde{\gamma}(\mathbf{k}) &= \tilde{C}(\mathbf{k})\psi(\mathbf{k})\psi(-\mathbf{k}). \end{aligned} \tag{3.34}$$

The analogy with (1.16) is now complete and we may use the analysis of Section IIB for the discussion of the small $\mathbf{k}$ behaviour. We shall do this in the next section for the more general case. It suffices here to point out that the singular part of $\tilde{G}(\mathbf{k})$ behaves at the fixed point as

$$\tilde{G}(\mathbf{k}) \sim \tilde{\chi}(\mathbf{k}) \sim |\mathbf{k}|^{d-2y_H}, \tag{3.35}$$

which implies for η (see (3.12)) the scaling relation

$$\eta = d + 2 - 2y_H. \tag{3.36}$$

This is a hyperscaling law equivalent to the relation

$$\delta = (d + 2 - \eta)/(d - 2 + \eta) \tag{3.37}$$

due to Fisher (1969).

C. General discussion of the correlation functions

For a generalization of the previous discussion, which hinged on the simplifications (3.18) and (3.25), we have to generalize the idea of scaling to spatially dependent interactions. Once this has been achieved the treatment follows rather closely the lines of the simplified version IIB.

Instead of considering the special magnetic perturbation (3.14) we employ the general expression

$$\delta\mathscr{H}(s) = \sum_a \delta K_a s_a. \tag{3.38}$$

By differentiation of the excess free energy δF with respect to δK_a and δK_b (setting $\delta\mathscr{H} = 0$ afterwards) we obtain the general correlation function

$$G_{ab} = \delta^2 F/\delta K_a \delta K_b = \langle s_a s_b\rangle - \langle s_a\rangle\langle s_b\rangle. \tag{3.39}$$

By taking a and b equal to the sites i and j one recovers the correlation function $G(\mathbf{r}_i - \mathbf{r}_j)$.

Applying the renormalization transformation (1.11) to $\mathscr{H} + \delta\mathscr{H}$ yields an excess $\delta C + \delta\mathscr{H}'$ where $\delta\mathscr{H}'$ will now (instead of (3.18)) be represented by the general expression

$$\delta\mathscr{H}'(s) = \sum_{a'} \delta K'_{a'} s'_{a'}. \tag{3.40}$$

Again the excess free energy fulfils the relation (cf. (3.19))

$$\delta F = \delta C + \delta F' \tag{3.41}$$

from which the following relation between the correlation functions can be derived by differentiating with respect to δK_a and δK_b

$$G_{ab} = C_{ab} + \sum_{a'}\sum_{b'} G'_{a'b'}(\delta K'_{a'}/\delta K_a)(\delta K'_{b'}/\delta K_b) + \sum_{a'}(\delta F'/\delta K'_{a'})(\delta^2 K'_{a'}/\delta K_a \delta K_b) \tag{3.42}$$

with the definition

$$C_{ab} = \delta^2 C/\delta K_a \delta K_b. \tag{3.43}$$

The other derivatives in (3.42) will be denoted by

$$T_{a'a} = \delta K'_{a'}/\delta K_a, \qquad T_{a',ab} = \delta^2 K'_{a'}/\delta K_a \delta K_b \tag{3.44}$$

and follow from the constraint averages

$$\langle s_a\rangle_{s'} = \delta C/\delta K_a + \sum_{a'} s'_{a'} T_{a'a}$$

$$\langle s_a s_b\rangle_{s'} - \langle s_a\rangle_{s'}\langle s_b\rangle_{s'} = C_{ab} + \sum_{a'} s'_{a'} T_{a',ab}. \tag{3.45}$$

$T_{a'a}$ is a generalization of $T_{\alpha'\alpha}$ and yields $T_{\alpha'\alpha}$ upon summation over a, as follows from (2.4)

$$\sum_{a\in\alpha} T_{a'a} = T_{\alpha'\alpha}, \tag{3.46}$$

where α' is the class to which a' belongs. In contrast to the previous discussion we may not drop $T_{a',ab}$. We shall incorporate the additional term in (3.42) involving $T_{a',ab}$, by taking suitable combinations of the δK_a as new coordinates (scaling fields) for which the second derivatives vanish. The new coordinates $u_i(\mathbf{k})$ depend on a wavenumber $\mathbf{k}$ and an index i giving the type of combination similarly to the index of the scaling fields. We will require that $u_i(\mathbf{k})$ transform as:

$$\delta u_i'(\mathbf{k}l) = l^{y_i}\delta u_i(\mathbf{k}) \tag{3.47}$$

where y_i is the same as the exponent of the scaling field u_i defined earlier. For the formation of the $\delta u_i(\mathbf{k})$ we consider first the fourier transforms†

$$\begin{aligned}\delta\tilde{K}_\alpha(\mathbf{k}) &= N^{-1}\sum_{a\in\alpha}\delta K_a \exp i\mathbf{k}\cdot\mathbf{r}_a,\\ \delta K_a &= \sum_{\mathbf{k}}\delta\tilde{K}_\alpha(\mathbf{k})\exp -i\mathbf{k}\cdot\mathbf{r}_a\end{aligned} \tag{3.48}$$

where $\mathbf{r}_a$ is the position of set a on the site lattice and where the summation over $\mathbf{k}$ runs through all wave vectors fitting the periodic boundary conditions of the site lattice. Then we construct in analogy of (3.27) a $\mathbf{k}$-dependent matrix

$$\tilde{T}_{\alpha\beta}(\mathbf{k}) = \sum_{b\in\beta} T_{a'b}\exp i\mathbf{k}\cdot(\mathbf{r}_{a'} - \mathbf{r}_b) \tag{3.49}$$

In view of (3.46) we have $T_{\alpha\beta}(\mathbf{0}) = T_{\alpha\beta}$. With (3.49) we derive the (first order) transformation law for the fourier coefficients δK_α as (see also appendix A)

$$\begin{aligned}\delta\tilde{K}_\alpha'(\mathbf{k}') &= (N')^{-1}\sum_{a'\in\alpha}\delta K_{a'}'\exp i\mathbf{k}'\cdot\mathbf{r}_{a'}'\\ &= (N')^{-1}\sum_{\mathbf{k}}\sum_{a'\in\alpha}\left\{\sum_\beta\tilde{T}_{\alpha\beta}(\mathbf{k})\delta\tilde{K}_\beta(\mathbf{k})\right\}\exp i[\mathbf{k}'\cdot\mathbf{r}_{a'}' - \mathbf{k}\cdot\mathbf{r}_{a'}].\end{aligned} \tag{3.50}$$

In the sum over a', it should be remembered that $\mathbf{r}_{a'}'$ measures the position of set a' (of cells) in the cell system while $\mathbf{r}_{a'}$ gives the position of a' in the site system. So $\mathbf{r}_{a'} = l\mathbf{r}_{a'}'$ and thus

$$(N')^{-1}\sum_{a'\in\alpha}\exp i[\mathbf{k}'\cdot\mathbf{r}_{a'}' - \mathbf{k}\cdot\mathbf{r}_{a'}] = \delta_{\mathbf{k}',\mathbf{k}l}. \tag{3.51}$$

† In order not to complicate the notation we change in the following the interpretation of α from being the class of all sets a which can be identified with each other by a symmetry operation on the lattice to the class of sets a which can be identified by a translation.

Therefore (3.50) reads

$$\delta\tilde{K}'_\alpha(\mathbf{k}l) = \sum_\beta \tilde{T}_{\alpha\beta}(\mathbf{k})\delta\tilde{K}_\beta(\mathbf{k}). \tag{3.52}$$

Now if we define a set of regular functions $\varphi^i_\alpha(\mathbf{k})$ through the relation

$$\sum_\alpha \varphi^{i'}_\alpha(\mathbf{k}l)\tilde{T}_{\alpha\beta}(\mathbf{k}) = l^{y_i}\varphi^i_\beta(\mathbf{k}) \tag{3.53}$$

then the combinations

$$\delta u_i(\mathbf{k}) = \sum_\alpha \varphi^i_\alpha(\mathbf{k})\delta\tilde{K}_\alpha(\mathbf{k}) \tag{3.54}$$

fulfil the transformation law (3.47). Equation (3.53) is the generalization of (3.32) but now referring to left eigenvectors. At the fixed point and for $\mathbf{k} = 0$, (3.53) reduces to

$$\sum_\alpha \varphi^{i*}_\alpha(\mathbf{0})T^*_{\alpha\beta} = l^{y_i}\varphi^{i*}_\beta(\mathbf{0}) \tag{3.55}$$

such that the values $\varphi^{i*}_\alpha(\mathbf{0})$ can be chosen to be equal to the left eigenvectors φ^i_α defined in (1.28). With this boundary condition the functions $\varphi^i_\alpha(\mathbf{k})$ may be obtained for non-zero $\mathbf{k}$ and in the neighbourhood of the fixed point by making a power series expansion of (3.53) in $\mathbf{k}$ and the scaling fields u_i. By repeated use of (3.53) one may solve for $\varphi^i_\alpha(\mathbf{k})$ for larger values of $\mathbf{k}$ and further away from the fixed point.

In order to change back and forth from the $\delta u_i(\mathbf{k})$ to the $\delta\tilde{K}_\alpha(\mathbf{k})$ we need also the right eigenvector version of (3.53)

$$\sum_\beta \tilde{T}_{\alpha\beta}(\mathbf{k})\psi^i_\beta(\mathbf{k}) = l^{y_i}\psi^{i'}_\alpha(\mathbf{k}l) \tag{3.56}$$

which defines the functions $\psi^i_\alpha(\mathbf{k})$ starting from the $\psi^{i*}_\alpha(\mathbf{0}) = \psi^i_\alpha$ defined as the right eigenvectors of $T^*_{\alpha\beta}$. By multiplying (3.56) on the left by $\varphi^{j'}_\alpha(\mathbf{k}l)$ and summing over α we find with (3.53)

$$l^{y_j}\sum_\alpha \varphi^j_\alpha(\mathbf{k})\psi^i_\alpha(\mathbf{k}) = l^{y_i}\sum_\alpha \varphi^{j'}_\alpha(\mathbf{k}l)\psi^{i'}_\alpha(\mathbf{k}l). \tag{3.57}$$

Since the φ^i_α and the ψ^i_α can be chosen orthogonal at the fixed point for $\mathbf{k} = 0$, this orthogonality propagates by (3.57) to $\mathbf{k} \neq 0$ and away from the fixed point:

$$\sum_\alpha \varphi^j_\alpha(\mathbf{k})\psi^i_\alpha(\mathbf{k}) = \delta_{ij}. \tag{3.58}$$

The completeness of the φ^i_α and ψ^i_α then also carries over from the fixed point and $\mathbf{k} = 0$ to

$$\sum_i \psi^i_\alpha(\mathbf{k})\varphi^i_\beta(\mathbf{k}) = \delta_{\alpha\beta}. \tag{3.59}$$

With (3.59) one can invert (3.54) to

$$\delta\tilde{K}_\alpha(\mathbf{k}) = \sum_i \psi^i_\alpha(\mathbf{k})\delta u_i(\mathbf{k}). \tag{3.60}$$

The next step would be to extend (3.54) to second order in δK_α in order to take the second-order derivatives in the last term in (3.42) into account. The extension should be such that (3.47) remains true to second order. We leave this point for the moment and investigate first the merits of having constructed scaling coordinates obeying (3.47). We define new correlation functions $\tilde{\chi}_{ij}(\mathbf{k})$ as

$$\tilde{\chi}_{ij}(\mathbf{k}) = \delta^2 f/\delta u_i(-\mathbf{k})\delta u_j(\mathbf{k}) \tag{3.61}$$

and a short-range correlation function $\tilde{\gamma}_{ij}(\mathbf{k})$ ($c = C/N$)

$$\tilde{\gamma}_{ij}(\mathbf{k}) = \delta^2 c/\delta u_i(-\mathbf{k})\delta u_j(\mathbf{k}) \tag{3.62}$$

From (3.41) and (3.47) it follows that they are related as

$$\tilde{\chi}_{ij}(\mathbf{k}) = \tilde{\gamma}_{ij}(\mathbf{k}) + l^{-d+y_i+y_j}\tilde{\chi}'_{ij}(\mathbf{k}l) \tag{3.63}$$

This is obviously a transformation law for correlation functions of the desired form. Note that in deriving (3.63) we have used the fact that new and old scaling coordinates are related linearly.

Now two points remain to be dealt with: the solution of $\tilde{\chi}_{ij}(\mathbf{k})$ in terms of $\tilde{\gamma}_{ij}(\mathbf{k})$ (for which the machinery developed in section IIC can be used), and the connection between $\tilde{\chi}_{ij}$ and G_{ab} on the one hand and $\tilde{\gamma}_{ij}$ and C_{ab} on the other hand. For the latter point we do need the second-order extension of (3.54) and (3.60). In Appendix A the straightforward but rather involved formulae are given. We present here the result.

Consider the fourier transforms of G_{ab} and C_{ab}

$$\begin{aligned}\tilde{G}_{\alpha\beta}(\mathbf{k}) &= \sum_{b\in\beta} G_{ab} \exp i\mathbf{k}\cdot(\mathbf{r}_a - \mathbf{r}_b)\\ \tilde{C}_{\alpha\beta}(\mathbf{k}) &= \sum_{b\in\beta} C_{ab} \exp i\mathbf{k}\cdot(\mathbf{r}_a - \mathbf{r}_b).\end{aligned} \tag{3.64}$$

Then $\tilde{G}_{\alpha\beta}(\mathbf{k})$ and $\tilde{\chi}_{ij}(\mathbf{k})$ are related through the formula

$$\tilde{G}_{\alpha\beta}(\mathbf{k}) = \sum_{ij} \varphi^i_\alpha(-\mathbf{k})\varphi^j_\beta(\mathbf{k})\tilde{\chi}_{ij}(\mathbf{k}) + \sum_\gamma (\partial f/\partial K_\gamma)\sum_i \psi^i_\gamma(\mathbf{0})\varphi^i_{\alpha\beta}(-\mathbf{k},\mathbf{k}). \tag{3.65}$$

Here $\varphi^i_{\alpha\beta}(\mathbf{k}_1,\mathbf{k}_2)$ is a regular function defined in appendix (A.7). Thus the singularities of $G_{\alpha\beta}(\mathbf{k})$ in the $\mathbf{k}$ variable are completely contained in $\tilde{\chi}_{ij}(\mathbf{k})$. The short-range part of the correlation function contains also some thermodynamic singularities through the appearance of $\partial f/\partial K_\alpha$ in the last term.

The connection between $\tilde{\gamma}_{ij}(\mathbf{k})$ and $\tilde{C}_{\alpha\beta}(\mathbf{k})$ is given by

$$\tilde{\gamma}_{ij}(\mathbf{k}) = \sum_{\alpha\beta} \tilde{C}_{\alpha\beta}(\mathbf{k})\psi^i_\alpha(-\mathbf{k})\psi^j_\beta(\mathbf{k}) - \sum_\alpha (\partial g/\partial K_\alpha) \sum_l \varphi^l_\alpha(\mathbf{0})\varphi^{lij}(-\mathbf{k}, \mathbf{k}). \tag{3.66}$$

The function $\varphi^{lij}(\mathbf{k}_1, \mathbf{k}_2)$ is regular and defined in appendix (A.10). So $\tilde{\gamma}_{ij}(\mathbf{k})$ is indeed short-ranged and regular in the thermodynamic variables.

The line of calculation is now as follows. First compute for a certain transformation the regular functions $\varphi^i_\alpha(\mathbf{k})$, $\psi^i_\alpha(\mathbf{k})$, $\varphi^i_{\alpha\beta}(\mathbf{k}, -\mathbf{k})$ and $\varphi^{lij}(-\mathbf{k}, \mathbf{k})$ as well as from (3.45) and (3.64) the $\tilde{C}_{\alpha\beta}(\mathbf{k})$ such that $\tilde{\gamma}_{ij}(\mathbf{k})$ can be calculated. Then solve (3.63) by the methods of Section IIC yielding $\tilde{\chi}_{ij}(\mathbf{k})$, which gives with (3.65) the desired $\tilde{G}_{\alpha\beta}(\mathbf{k})$. Of this process we outline the determination of the long-range behaviour of the correlation functions.

At criticality the singular part in $\mathbf{k}$ of $\tilde{\chi}_{ij}(\mathbf{k})$ is given by

$$[\tilde{\chi}_{ij}(\mathbf{k})]_{\text{sing}} = \tilde{A}_{ij}(\hat{\mathbf{k}})|\mathbf{k}|^{d-y_i-y_j}, \tag{3.67}$$

and the amplitude $\tilde{A}_{ij}(\hat{\mathbf{k}})$ given by

$$\tilde{A}_{ij}(\hat{\mathbf{k}}) = [\log l]^{-1} \int_0^\infty \mathrm{d}t\, t^{-1-d+y_i+y_j}[\tilde{\gamma}_{ij}(\hat{\mathbf{k}}t)]_{\text{rem}} \tag{3.68}$$

where the subscript "rem" is defined as in Section IIC: subtract the powers k^p from $\tilde{\gamma}_{ij}(\mathbf{k})$ for which $l^p < l^{d-y_i-y_j}$. (For the irrelevant eigenvalues $y_i < 0$, $y_j < 0$, now also a remainder and thus singular parts and amplitudes exist.) The behaviour (3.67) implies that the exponent η_{ij} for the correlation function $\tilde{\chi}_{ij}$ is given by (cf. (3.36))

$$\eta_{ij} = d + 2 - y_i - y_j. \tag{3.69}$$

The behaviour near criticality will be described by taking one relevant scaling field u_T into account, which couples to the temperature difference with the critical point. Using expression (2.41) we have

$$[\tilde{\chi}_{ij}(u_T, \mathbf{k})]_{\text{sing}} = |\mathbf{k}|^{\eta_{ij}-2}\tilde{A}_{ij}(\hat{\mathbf{k}}; u_T|\mathbf{k}|^{-y_T}) \tag{3.70}$$

with the amplitude given by the integral

$$\tilde{A}_{ij}(\hat{\mathbf{k}}; v_T) = [\log l]^{-1} \int_0^\infty \mathrm{d}t\, t^{-1-d+y_i+y_j}[\tilde{\gamma}_{ij}(\hat{\mathbf{k}}t; v_T t^{y_T})]_{\text{rem}}. \tag{3.71}$$

For the spatial behaviour (3.70) implies

$$\tilde{\chi}_{ij}(\mathbf{r}) \simeq \frac{A_{ij}(\hat{\mathbf{r}}; ru_T^{1/y_T})}{r^{d-2+\eta_{ij}}}. \tag{3.72}$$

Thus a correlation length ξ

$$\xi = u_T^{-1/y_T} \tag{3.73}$$

appears, with exponent $\nu = 1/y_T$ (cf. (3.13)), which is the same for all correlation functions.

For the correlation function $G_{\alpha\beta}(\mathbf{r})$ the result (3.67) yields at criticality

$$G_{\alpha\beta}(\mathbf{r}) \simeq \sum_{ij} \phi^i_\alpha \phi^j_\beta A_{ij}(\hat{\mathbf{r}})/r^{(d-2+\eta_{ij})}. \tag{3.74}$$

As we are interested in the dominant decay we could drop the k-dependence in $\varphi^i_\alpha(\mathbf{k})$ in (3.65). For the same reason in the summation in (3.74) over i and j one should take only the term with the *lowest* η_{ij} for which the coefficient in front does not vanish; e.g. for α and β both referring to single sites $G_{\alpha\beta}(\mathbf{r})$ couples to the combination i, j both equal to the relevant (odd) magnetic mode H.

The case where α and β both refer to nearest-neighbour pair (the energy–energy correlation function for the Ising model) couples to i, j both equal to the relevant (even) temperature mode T.

D. Connection between exponents and eigenvectors

After having analysed the general behaviour of the correlation function we come back to the relation (3.4) for the correlation function which we generalize to:

$$g'_{\alpha\beta}(\mathbf{r}'_{i'} - \mathbf{r}'_{j'}) = \sum_{\{s'\}} s'_{\alpha,i'} s'_{\beta,j'} \langle P(s', s)\rangle. \tag{3.75}$$

Rather than trying to present general formulae for arbitrary $P(s', s)$ we illustrate the content of (3.75) for the triangular (non-linear) weight-factor (1.18)). First we study the case of the pair correlation function where α and β both stand for "single site". Then the sum over (3.75) selects out of the product for $P(s', s)$ the factors referring to cells i' and j'. Instead of (3.7) we obtain

$$g'(\mathbf{r}'_{i'} - \mathbf{r}'_{j'}) = \tfrac{1}{4}\langle (s^1_{i'} + s^2_{i'} + s^3_{i'} - s^1_{i'} s^2_{i'} s^3_{i'})(s^1_{j'} + s^2_{j'} + s^3_{j'} - s^1_{j'} s^2_{j'} s^3_{j'})\rangle. \tag{3.76}$$

Now there appear on the right-hand side singlet–singlet, singlet–triplet and triplet–triplet correlation functions. Writing $\alpha = s$ for a single site and $\alpha = t$ for a triplet and using (3.74) for the asymptotic behaviour we obtain at the fixed point

$$\sum_{i,j} \frac{\phi^i_s \phi^j_s A_{ij}}{r'^{d-2+\eta_{ij}}} = \tfrac{1}{4} \sum_{i,j} \frac{(3\phi^i_s - \phi^i_t)(3\phi^j_s - \phi^j_t) A_{ij}}{(lr')^{d-2+\eta_{ij}}}. \tag{3.77}$$

The leading terms on both sides have to be equal. For the odd $\alpha = s, t$ the slowest decay comes from the magnetic (odd) eigenvalue y_H. So the terms with $i = j = H$ must be equal on both sides in (3.77), which is the case of

$$l^{d-y_H} \phi^H_s = \tfrac{3}{2}\phi^H_s - \tfrac{1}{2}\phi^H_t \tag{3.78}$$

where we used (3.69) for $\eta_{HH} = d + 2 - 2y_H$.

Equation (3.78) is a direct connection between the exponent y_H and the components ϕ_α^H of the (left) eigenvector. It replaces the relation (3.11) which was derived for the linear weight-factor (3.6). The numbers $\frac{3}{2}$ and $\frac{1}{2}$ occurring at the right-hand side of (3.78) are typical for the triangular weight-factor (1.18). Since (3.77) includes only the *dominant* decay for each mode, we may not draw further conclusions for the other modes in (3.77). Kadanoff and Houghton (1975) have used a relation equivalent to (3.78) to optimize a free parameter in their renormalization transformation. (See Section V.B.3).

Relations like (3.78) can be derived for other choices of α and β in (3.75), but the equation soon becomes of little practical value. By taking α and β to be a pair of neighbouring cells ($\alpha = \beta = p$) one can derive a relation between the even relevant exponent y_T and the corresponding eigenvector φ_α^T of the type

$$l^{d-y_T}\phi_p^T = \sum_\alpha w_\alpha \phi_\alpha^T. \tag{3.79}$$

where the w_α are numbers involving nearest-neighbour pairs ($\alpha = p$), other more distant pairs, quartets and a sextet of sites (e.g. $w_p = \frac{1}{2}$).

IV. Computational Methods

The crux of the renormalization transformation (1.22) is to evaluate the functions $K'(K)$ to a sufficient accuracy, such that reliable conclusions can be drawn from them. To that purpose the summation over the site spin configurations $\{s\}$ in (1.11) has to be carried out. $\mathcal{H}(s)$ is in general more complicated than e.g. a "pure" Ising Hamiltonian. Moreover the summation over $\{s\}$ is constrained by the weight-factor $P(s', s)$, so its exact evaluation is, except for a few cases, hardly feasible. Also, it is not the purpose of the renormalization theory to add to the number of exactly solvable models, but rather to provide for more realistic cases a computational scheme for the critical singularities and the free energy.

In this Section we treat approximation schemes of general character, which are partly complementary. These schemes are more or less copies of comparable approximation schemes for the Ising model. Here we shall review briefly the main methods that have been used so far:

A. simple renormalization transformations on finite lattices;

B. the cumulant approximation;

C. the cluster approximation.

In all the applications we are going to deal with, the site lattice will be subdivided into cells, the cell lattice having the same symmetry (or approximately the same symmetry) as the underlying site lattice. For the triangular lattice a possible subdivision in cells has already been given in Fig. 1a. For