Definition A subset $S \subset \mathbb{R}$ is **compact** if whenever it is contained in the union of a family $\mathcal{F}$ of open sets, then it is contained in the union of some finite number of the sets in $\mathcal{F}$.

What does this mean? What would it mean for a set S to not be compact?

To show S is not compact, you would have show that there exists a family of open sets $\mathcal{F}$, whose union contains S, but S is not contained in the union of any finite number of the sets from $\mathcal{F}$.

Let $S = (0, 1]$. I claim that S is not compact.

Let $\mathcal{F}$ be the family $A_n = (1/n, 2)$. Then

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (1/n, 2) \supset (0, 1]$$

because for every $x \in (0, 1]$, we can find an n such that $1/n < x$, and so $x \in (1/n, 2)$. We want to show that S is not contained in any finite subcover. Assume that S is contained

$$S \subset \bigcup_{k \in I} A_k, \quad I \text{ a finite subset of } \mathbb{N}$$

Let N be the largest integer in I. Then A_N contains all the other A_k's for $k \in I$. Choose $x \in (0, 1]$ so that $x < 1/N$. Then $x \in S$, but $x \notin A_N$, so $x \notin A_k$ for any $k \in I$. So we have just proved that $(0, 1]$ is not a compact set.

What about $S = [0,1]$? Do the sets A_n cover this set? No, because $0 \in [0,1]$ but not in any of the A_n.

What if we add one more set to the family in order to cover 0? For example, we could take the union of $\mathcal{F}$ and

$$\{(-0.0000000001, 0.0000000001)\}.$$

Is there a finite subcover? Yes: we can cover $(0,1]$ with

$$(-0.0000000001, 0.0000000001)$$

and $(1/N, 2)$ where $N > 10^{10}$. In fact we shall see later that $[0,1]$ is compact.

A final example: $S = [0, \infty)$. This is not compact because the cover $\{(-1, n) : n \in \mathbb{N}\}$ has no finite subscover.