Lesson 1: What is a Differential Equation?

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Preliminaries

What's an Algebraic Equation?

An algebraic equation is a statement that two algebraic expressions are equal to each other. The algebraic expressions are built up from variables, and constants, using algebraic operations.

A solution to an equation is a set of values of the variables that makes the equation true.

What's a differential equation?

$$\begin{aligned} x' &= t^2 \\ y' &= 0.1y \\ x'' - 2x' - 3x &= 0 \\ y' &= x^2 + y^2 \\ xx' + 2x &= 3x'' \end{aligned}$$

An (ordinary) differential equation is an equation built up from constants, and independent variable, a dependent variable, using the operations of arithmetic and differentiation.

Solutions to differential equations

A solution to a differential equation with variables x and t is a function $x = f(t)$ so that when I substitute the derivative for x', etc., I get an equality of two functions.

For the first three equations we have solutions

$$\begin{aligned} x &= \frac{t^3}{3} + C \\ y &= e^{0.1(t+C)} \quad \text{or} \quad y = y_0 e^{0.1t}, \quad y_0 = e^{0.1C} \\ x &= Ae^{3t} + Be^{-t} \end{aligned}$$

The solutions e^{3t} and e^{-t} work in the third equation because

$$\begin{aligned} 9e^{3t} - 2(3e^{3t}) - 3e^{3t} &= (3^2 - 2(3) - 3)e^{3t} = 0 \\ (-1)^2 e^{-t} - 2(-e^{-t}) - 3e^{-t} &= ((-1)^2 + 2 - 3)e^{-t} = 0. \end{aligned}$$

Where do differential equations come from?

The rate of change of a population is proportional to the population. For example, if $y' = 0.1y$ then the population is growing at a rate of 10% per unit time. The general law is

$$y' = ky$$

Here k is the annual growth rate. If the growth rate is 10% a year, then what are the solutions to this equation? This is the equation we looked at above, $y' = -0.1y$, and we guessed the solution $y = y_0 e^{0.1t}$. Here are the graphs of these solutions for $y_0 = 10, 20, 30$. What is the interpretation of y_0?

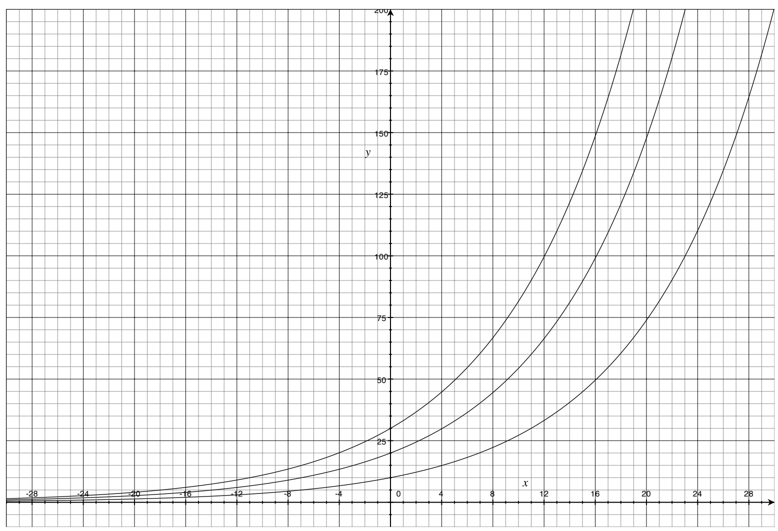

Figure 1: Graph of $y = y_0 e^{0.1t}$ for $y_0 = 10, 20, 30$

Suppose that the rate of change of a population is proportional to the population, and to the amount of room left. This leads to a different equation:

$$y' = ay(L - y)$$

It is difficult to guess the solution of this equation, but we can ask the computer to draw the graph of the solution. See Figure 2.

How do we find solutions?

- Integration (for $x' = t^2$)
- Guessing (for $y' = 0.1y$)

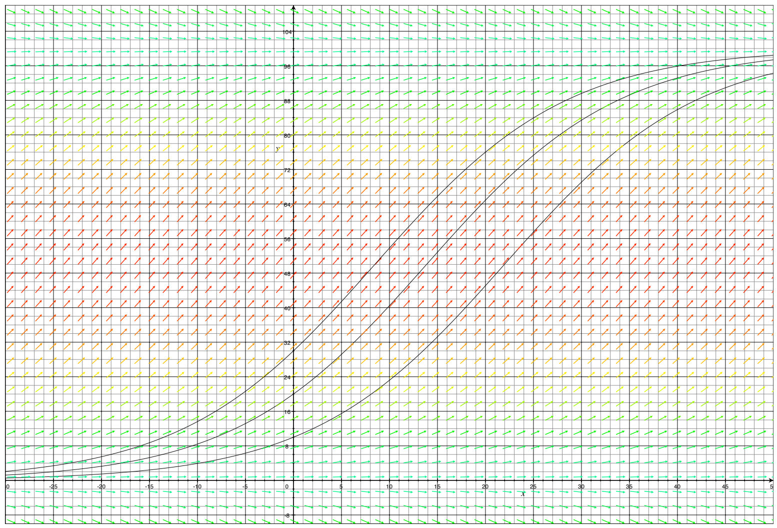

Figure 2: Graph of solutions for $y(0) = 10, 20, 30$

How might we solve $y' = 0.1y$ by integration?

$$\begin{aligned} y' &= 0.1y \\ \frac{y'}{y} &= 0.1 \\ \int \frac{y'}{y} dt = \ln(y) &= 0.1t + C \\ y &= e^{0.1t+C} \end{aligned}$$

- Computer (for $y' = 0.001(100 - y)$)

  How might the computer be drawing the solutions? Imagine the computer has drawn the graph to a certain point. What information does it need to keep drawing the graph? It needs to know what direction to point in. Since the slope is $0.001(100 - y)$, we can indicate this be drawing a line with this slope at the point (t, y).